1
Sets

INTRODUCTION

In our mathematical language, *everything in this universe, whether living or non-living, is called an object.*

A given collection of objects is said to be well defined, if we can definitely say whether a given particular object belongs to the collection or not.

SET *A well-defined collection of objects is called a set.*

The objects in a set are called its ***members*** *or* ***elements*** *or* ***points.***

We denote sets by capital letters *A, B, C, X, Y, Z,* etc.

If a is an element of a set A, we write, $a \in A$, which means that ***a belongs to A*** or that ***a is an element of A.***

If a does not belong to A, we write, $a \notin A$.

ILLUSTRATIONS

(i) The collection of all vowels in the English alphabet contains five elements, namely a, e, i, o, u.
So, this collection is well defined and therefore, it is a set.

(ii) The collection of all odd natural numbers less than 10 contains the numbers 1, 3, 5, 7, 9.
So, this collection is well defined and therefore, it is a set.

(iii) The collection of all prime numbers less than 20 contains the numbers 2, 3, 5, 7, 11, 13, 17, 19.
So, this collection is well defined and therefore, it is a set.

(iv) All possible roots of the quadratic equation $x^2 - x - 6 = 0$ are -2 and 3.
So, the collection of all possible roots of $x^2 - x - 6 = 0$ is well defined and therefore, it is a set.

(v) The collection of all rivers of India, is clearly well defined and therefore, it is a set.
Clearly, river Ganga belongs to this set while river Nile does not belong to it.

(vi) The collection of five most talented writers of India is not a set, since no rule has been given for deciding whether a given writer is talented or not.

(vii) The collection of most dangerous animals of the world is not a set, since no rule has been given for deciding whether a given animal is dangerous or not.

(viii) The collection of five most renowned mathematicians of the world is not a set, since there is no criterion for deciding whether a mathematician is renowned or not.

(ix) The collection of all beautiful girls of India is not a set, since the term '*beautiful*' is vague and it is not well defined.

Similarly, *'rich persons', 'honest persons', 'good players', 'old people', 'young men',* etc., do not form sets.

However, *'blind persons', 'dumb persons', 'illiterate persons', 'retired persons',* etc., form sets.

HOW TO DESCRIBE OR SPECIFY A SET?

There are two methods of describing a set.

I. ROSTER FORM, OR TABULATION METHOD *Under this method, we list all the members of the set within braces* { } *and separate them by commas.*

Note that the order in which the elements are listed, is immaterial.

EXAMPLES *Write each of the following sets in the roster form:*

(i) A = set of all factors of 24.

(ii) B = set of all prime numbers between 50 *and* 70.

(iii) C = set of all integers between $-\frac{3}{2}$ *and* $\frac{11}{2}$.

(iv) D = set of all consonants in the English alphabet which precede k.

(v) E = set of all letters in the word 'TRIGONOMETRY'.

(vi) F = set of all months having 30 *days.*

SOLUTION (i) All factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.

$\therefore$ $A = \{1, 2, 3, 4, 6, 8, 12, 24\}$.

(ii) All prime numbers between 50 and 70 are 53, 59, 61, 67.

$\therefore$ $B = \{53, 59, 61, 67\}$.

(iii) All integers between $-\frac{3}{2}$ and $\frac{11}{2}$ are $-1, 0, 1, 2, 3, 4, 5$.

$\therefore$ $C = \{-1, 0, 1, 2, 3, 4, 5\}$.

(iv) All consonants preceding k are b, c, d, f, g, h, j.

$\therefore$ $D = \{b, c, d, f, g, h, j\}$.

(v) It may be noted here that the repeated letters are taken only once each.

$\therefore$ $E = \{T, R, I, G, O, N, M, E, Y\}$.

(vi) We know that the months having 30 days are April, June, September, November.

$\therefore$ $F = \{April, June, September, November\}$.

NOTE We denote the sets of *all natural numbers*, *all integers*, *all rational numbers* and *all real numbers* by N, Z, Q and R respectively.

II. SET-BUILDER FORM *Under this method of describing a set, we list the property or properties satisfied by all the elements of the set.*

We write, $\{x : x \text{ has properties } P\}$.

We read it as, '*the set of all those x such that each x satisfies properties P*'.

ILLUSTRATIONS

EXAMPLE 1 *Write the set* $A = \{1, 2, 3, 4, 5, 6, 7\}$ *in the set-builder form.*

SOLUTION Clearly, A = set of all natural numbers less than 8.

Thus, in the set-builder form, we write it as

$$A = \{x : x \in N \text{ and } x < 8\}.$$

EXAMPLE 2 *Write the set* $B = \{1, 2, 4, 7, 14, 28\}$ *in the set-builder form.*

SOLUTION Clearly, B = set of all factors of 28.

Thus, in the set-builder form, we write it as

$$B = \{x : x \in N \text{ and } x \text{ is a factor of } 28\}.$$

EXAMPLE 3 *Write the set* $C = \{2, 4, 8, 16, 32\}$ *in the set-builder form.*

SOLUTION Clearly, $C = \{2^1, 2^2, 2^3, 2^4, 2^5\}$.

Thus, in the set-builder form, we write it as

$$C = \{x : x = 2^n, \text{ where } n \in N \text{ and } 1 \le n \le 5\}.$$

EXAMPLE 4 *Write the set* $D = \{-6, -4, -2, 0, 2, 4, 6\}$ *in the set-builder form.*

SOLUTION Clearly, D = set of even integers from -6 to 6.

Thus, in the set-builder form, we write it as

$$D = \{x : x = 2n, \text{ where } n \in Z \text{ and } -3 \le n \le 3\}.$$

EXAMPLE 5 *Write the set* $E = \{3, 6, 9, 12, 15, 18\}$ *in the set-builder form.*

SOLUTION Clearly, $E = \{3 \times 1, 3 \times 2, 3 \times 3, 3 \times 4, 3 \times 5, 3 \times 6\}$.

Thus, in the set-builder form, we write it as

$$E = \{x : x = 3n, \text{ where } n \in N \text{ and } 1 \le n \le 6\}.$$

EXAMPLE 6 *Write the set* $F = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}\right\}$ *in the set-builder form.*

SOLUTION Clearly, we have

$$F = \left\{x : x = \frac{n}{(n+1)}, \text{ where } n \in N \text{ and } 1 \le n \le 8\right\}.$$

EXAMPLE 7 *Write the set* $G = \{1, 3, 5, 7, 9, 11, \ldots\}$ *in the set-builder form.*

SOLUTION Clearly, G = set of all odd natural numbers.

Thus, in the set-builder form, we write it as

$$G = \{x : x \in N \text{ and } x \text{ is odd}\}.$$

EXAMPLE 8 *Write the set* $H = \{1, 4, 9, 16, 25, 36, \ldots\}$ *in the set-builder form.*

SOLUTION Clearly, H is the set of the squares of all natural numbers.

So, in the set-builder form, we write it as

$$H = \{x : x = n^2, \text{ where } n \in N\}.$$

EXAMPLE 9 *Match each of the sets on the left in the roster form with the same set on the right given in set-builder form:*

(i) {23, 29}	*(a)* $\{x : x = 3^n, n \in N \text{ and } 1 \le n \le 5\}$
(ii) {*B, E, T, R*}	*(b)* $\{x : x = n^3, n \in N \text{ and } 2 \le n \le 6\}$
(iii) {3, 9, 27, 81, 243}	*(c)* $\{x : x \text{ is prime}, 20 < x < 30\}$
(iv) {8, 27, 64, 125, 216}	*(d)* $\{x : x \text{ is a letter of the word 'BETTER'}\}$

SOLUTION (i) {23, 29} = set of prime numbers between 20 and 30

$= \{x : x \text{ is prime}, 20 < x < 30\}$.

$\therefore$ (i) $\leftrightarrow$ (c).

(ii) {B, E, T, R} = set of letters in the word 'BETTER'}

$= \{x : x \text{ is a letter in the word 'BETTER'}\}$.

$\therefore$ (ii) $\leftrightarrow$ (d).

(iii) $\{3, 9, 27, 81, 243\} = \{3^1, 3^2, 3^3, 3^4, 3^5\}$

$= \{x : x = 3^n, n \in N \text{ and } 1 \le n \le 5\}$.

$\therefore$ (iii) $\leftrightarrow$ (a).

(iv) $\{8, 27, 64, 125, 216\} = \{2^3, 3^3, 4^3, 5^3, 6^3\}$

$= \{x : x = n^3, n \in N, 2 \le n \le 6\}$.

$\therefore$ (iv) $\leftrightarrow$ (b).

EXERCISE 1A

1. Which of the following are sets? Justify your answer.
   (i) The collection of all whole numbers less than 10.
   (ii) The collection of good hockey players in India.
   (iii) The collection of all questions in this chapter.
   (iv) The collection of all difficult chapters in this book.
   (v) A collection of Hindi novels written by Munshi Prem Chand.
   (vi) A team of 11 best cricket players of India.
   (vii) The collection of all the months of the year whose names begin with the letter *M*.
   (viii) The collection of all interesting books.
   (ix) The collection of all short boys of your class.
   (x) The collection of all those students of your class whose ages exceed 15 years.
   (xi) The collection of all rich persons of Kolkata.

(xii) The collection of all persons of Kolkata whose assessed annual incomes exceed (say) ` 20 lakh in the financial year 2016–17.

(xiii) The collection of all interesting dramas written by Shakespeare.

2. Let A be the set of all even whole numbers less than 10.

(a) Write A in roster form.

(b) Fill in the blanks with the approximate symbol $\in$ or $\notin$:

(i) 0 ..... A (ii) 10 ..... A (iii) 3 ..... A (iv) 6 ..... A

3. Write the following sets in roster form:

(i) $A = \{x : x \text{ is a natural number}, 30 \le x < 36\}$.

(ii) $B = \{x : x \text{ is an integer and } -4 < x < 6\}$.

(iii) $C = \{x : x \text{ is a two-digit number such that the sum of its digits is 9}\}$.

(iv) $D = \{x : x \text{ is an integer}, x^2 \le 9\}$.

(v) $E = \{x : x \text{ is a prime number, which is a divisor of 42}\}$.

(vi) $F = \{x : x \text{ is a letter in the word 'MATHEMATICS'}\}$.

(vii) $G = \{x : x \text{ is a prime number and } 80 < x < 100\}$.

(viii) $H = \{x : x \text{ is a perfect square and } x < 50\}$.

(ix) $J = \{x : x \in R \text{ and } x^2 + x - 12 = 0\}$.

(x) $K = \{x : x \in N, x \text{ is a multiple of 5 and } x^2 < 400\}$.

4. List all the elements of each of the sets given below:

(i) $A = \{x : x = 2n, n \in N \text{ and } n \le 5\}$.

(ii) $B = \{x : x = 2n + 1, n \in W \text{ and } n < 5\}$.

(iii) $C = \left\{x : x = \frac{1}{n}, n \in N \text{ and } n < 6\right\}$.

(iv) $D = \{x : x = n^2, n \in N \text{ and } 2 \le n \le 5\}$.

(v) $E = \{x : x \in Z \text{ and } x^2 = x\}$.

(vi) $F = \left\{x : x \in Z \text{ and } -\frac{1}{2} < x < \frac{13}{2}\right\}$.

(vii) $G = \left\{x : x = \frac{1}{(2n-1)}, n \in N \text{ and } 1 \le n \le 5\right\}$.

(viii) $H = \{x : x \in Z, |x| \le 2\}$.

5. Write each of the sets given below in set-builder form:

(i) $A = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}\right\}$ (ii) $B = \left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \frac{6}{37}, \frac{7}{50}\right\}$

(iii) $C = \{53, 59, 61, 67, 71, 73, 79\}$ (iv) $D = \{-1, 1\}$

(v) $E = \{14, 21, 28, 35, 42, \ldots, 98\}$

6. Match each of the sets on the left described in roster form with the same set on the right described in the set-builder form:

(i) {–5, 5} (a) $\{x : x \in Z \text{ and } x^2 < 16\}$

(ii) {1, 2, 3, 6, 9, 18} (b) $\{x : x \in N \text{ and } x^2 = x\}$

(iii) $\{-3, -2, -1, 0, 1, 2, 3\}$ (c) $\{x : x \in Z \text{ and } x^2 = 25\}$

(iv) {P, R, I, N, C, A, L} (d) $\{x : x \in N \text{ and } x \text{ is a factor of } 18\}$

(v) {1} (e) {x : x is a letter in the word 'PRINCIPAL'}

ANSWERS (EXERCISE 1A)

1. (i), (iii), (v), (vii), (x), (xii)

2. (a) $A = \{0, 2, 4, 6, 8\}$ (b) (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\in$

3. (i) $A = \{30, 31, 32, 33, 34, 35\}$ (ii) $B = \{-3, -2, -1, 0, 1, 2, 3, 4, 5\}$

(iii) $C = \{18, 81, 27, 72, 36, 63, 45, 54, 90\}$

(iv) $D = \{-3, -2, -1, 0, 1, 2, 3\}$ (v) $E = \{2, 3, 7\}$

(vi) $F = \{M, A, T, H, E, I, C, S\}$ (vii) $G = \{83, 89, 97\}$

(viii) $H = \{1, 4, 9, 16, 25, 36, 49\}$ (ix) $J = \{-4, 3\}$

(x) $K = \{5, 10, 15\}$

4. (i) $A = \{2, 4, 6, 8, 10\}$ (ii) $B = \{1, 3, 5, 7, 9\}$

(iii) $C = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ (iv) $D = \{4, 9, 16, 25\}$

(v) $E = \{0, 1\}$ (vi) $F = \{0, 1, 2, 3, 4, 5, 6\}$

(vii) $G = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\right\}$ (viii) $H = \{-2, -1, 0, 1, 2\}$

5. (i) $A = \left\{x : x = \frac{1}{n^2}, n \in N \text{ and } 1 \le n \le 7\right\}$

(ii) $B = \left\{x : x = \frac{n}{(n^2+1)}, n \in N \text{ and } 1 \le n \le 7\right\}$

(iii) $C = \{x : x \text{ is prime}, 50 < x < 80\}$

(iv) $D = \{x : x \in Z, x^2 = 1\}$

(v) $E = \{x : x = 7n, n \in N, 2 \le n \le 14\}$

6. (i) $\leftrightarrow$ (c), (ii) $\leftrightarrow$ (d), (iii) $\leftrightarrow$ (a), (iv) $\leftrightarrow$ (e), (v) $\leftrightarrow$ (b)

HINTS TO SOME SELECTED QUESTIONS

1. (i) Clearly, all whole numbers less than 10 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

(ii) Clearly, there is no specific criterion to decide whether a given hockey player of India, is good or not.

So, the given collection is not a set.

(iii) The collection of all questions in this chapter is a set, because, if a question is given then we can easily decide whether it is a question of the chapter or not.

(iv) Clearly, the term '*difficult*' is vague. So, the given collection is not well defined and therefore, it is not a set.

(v) Suppose we are given a collection of Hindi novels written by Munshi Prem Chand. Now, if we take any Hindi novel then we can clearly decide whether it belongs to our collection or not.

So, the given collection is a set.

(vi) The term '*best*' is vague. So, the given collection is not a set.

(vii) The given collection has definite members, namely March and May. So, this collection is a set.

(viii) The term '*interesting*' is vague. So, the given collection is not a set.

(ix) The term '*short*' is vague. So, the given collection is not a set.

(x) Clearly, it contains definite members. So, it is a set.

(xi) The term '*rich*' is vague. So, the given collection is not a set.

(xii) The given collection is clearly well defined. So, it is a set.

(xiii) The term '*interesting*' is vague. So, the given collection is not a set.

SOME TERMS RELATED TO SETS

EMPTY SET *A set containing no element at all is called* ***the empty set,*** *or* ***the null set,*** *or* ***the void set,*** denoted by ϕ, or $\{\}$.

A set which has at least one element is called a ***nonempty set.***

Examples of empty sets

(i) $\{x : x \in N \text{ and } 2 < x < 3\} = \phi$, since there is no natural number lying between 2 and 3.

(ii) $\{x : x \text{ is a number}, x \neq x\} = \phi$, since there is no number which is not equal to itself.

(iii) $\{x : x \in N, x < 5 \text{ and } x > 7\} = \phi$, since there is no natural number which is less than 5 and greater than 7.

(iv) $\{x : x \in R \text{ and } x^2 = -1\} = \phi$, since there is no real number whose square is -1.

(v) $\{x : x \text{ is rational and } x^2 - 2 = 0\} = \phi$, since there is no rational number whose square is 2.

(vi) $\{x : x$ is an even prime number greater than 2$\} = \phi$, since there is no prime number which is even and greater than 2.

(vii) $\{x : x$ is a point common to two parallel lines$\} = \phi$, since there is no point common to two parallel lines.

SINGLETON SET *A set containing exactly one element is called a singleton set.*

Examples of singleton sets

(i) $\{0\}$ is a singleton set whose only element is 0.

(ii) $\{15\}$ is a singleton set whose only element is 15.

(iii) $\{-8\}$ is a singleton set whose only element is -8.

(iv) $\{x : x \in N \text{ and } x^2 = 4\} = \{2\}$, since 2 is the only natural number whose square is 4.

However, $\{x : x \in Z \text{ and } x^2 = 4\} = \{-2, 2\}$, which is not a singleton set.

(v) Consider the set $\{x : x \in R \text{ and } x^3 - 1 = 0\}$.

Now, $x^3 - 1 = 0 \Rightarrow (x-1)(x^2 + x + 1) = 0$

$$\Rightarrow x = 1 \text{ or } x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

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Thus, the given equation has one real root, namely $x = 1$.

$\therefore \quad \{x : x \in R \text{ and } x^3 - 1 = 0\} = \{1\}$, which is a singleton set.

FINITE AND INFINITE SETS

An empty set or a nonempty set in which the process of counting of elements surely comes to an end is called a finite set. A set which is not finite is called an infinite set.

The number of distinct elements contained in a finite set A is denoted by $n(A)$.

Examples of finite sets

(i) Let $A = \{2, 4, 6, 8, 10, 12\}$.
Then, A is clearly a finite set and $n(A) = 6$.

(ii) Let B = set of all letters in the English alphabet.
Then, $n(B) = 26$ and therefore, B is finite.

(iii) Let $C = \{x : x \in Z \text{ and } x^2 - 36 = 0\}$.
Then, $C = \{-6, 6\}$, which is clearly a finite set and $n(C) = 2$.

(iv) The set of all persons on earth is a finite set.

(v) The set of all animals on earth is a finite set.

Examples of infinite sets

(i) The set of all points on the arc of a circle is an infinite set.

(ii) The set of all points on a line segment is an infinite set.

(iii) The set of all circles passing through a given point is an infinite set.

(iv) The set of all straight lines parallel to a given line, say the x-axis, is an infinite set.

(v) The set of all positive integral multiples of 5 is an infinite set.
Let Z^+ be the set of all positive integers.
Then, $\{5x : x \in Z^+\} = \{5, 10, 15, 20, 25, \ldots\}$ is an infinite set.

(vi) Each of the sets N, Z, Q and R is an infinite set.

NOTE All infinite sets cannot be described in roster form.
For example, the set R of all real numbers cannot be described in this form, since the elements of this set do not follow a particular pattern.

EQUAL SETS *Two nonempty sets A and B are said to be equal, if they have exactly the same elements and we write, $A = B$.*

Otherwise, the sets are unequal and we write, $A \neq B$.

REMARKS (i) The elements of a set may be listed in any order.
Thus, {1, 2, 3} = {2, 1, 3} = {3, 2, 1}.

(ii) The repetition of elements in a set has no meaning.
Thus, {1, 1, 2, 2, 3} = {1, 2, 3}.

Some examples of equal sets

EXAMPLE 1 *Let A = set of letters in the word 'follow',*
and B = set of letters in the word 'wolf'.
Show that $A = B$.

SOLUTION Clearly, we have

$A = \{f, o, l, w\}$ and $B = \{w, o, l, f\}$.

Clearly, A and B have exactly same elements.

$\therefore \quad A = B.$

EXAMPLE 2 *Let $A = \{p, q, r, s\}$ and $B = \{q, r, p, s\}$. Are A and B equal?*

SOLUTION Since A and B have exactly the same elements, so $A = B$.

EXAMPLE 3 *Show that ϕ, $\{0\}$ and 0 are all different.*

SOLUTION We know that ϕ is a set containing no element at all.

And, $\{0\}$ is a set containing one element, namely 0.

Also, 0 is a number, not a set.

Hence, ϕ, $\{0\}$ and 0 are all different.

EXAMPLE 4 *Let $A = \{x : x \in N, x^2 - 9 = 0\}$ and $B = \{x : x \in Z, x^2 - 9 = 0\}$.*
Show that $A \neq B$.

SOLUTION $x^2 - 9 = 0 \Rightarrow (x+3)(x-3) = 0 \Rightarrow x = -3$ or $x = 3$.

$\therefore \quad A = \{x : x \in N, x^2 - 9 = 0\} = \{3\}$ $[\because -3 \notin N]$

and $B = \{x : x \in Z, x^2 - 9 = 0\} = \{-3, 3\}$.

Hence, $A \neq B$.

EQUIVALENT SETS *Two finite sets A and B are said to be equivalent, if $n(A) = n(B)$.*

Equal sets are always equivalent. But, equivalent sets need not be equal.

EXAMPLE 1 Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$.

Then, $n(A) = n(B) = 3$.

So, A and B are equivalent.

Clearly, $A \neq B$.

Hence, A and B are equivalent sets but not equal.

EXAMPLE 2 *Show that $\{0\}$ and ϕ are not equivalent sets.*

SOLUTION Let $A = \{0\}$ and $B = \phi$.

Then, clearly $n(A) = 1$ and $n(B) = 0$.

$\therefore \quad n(A) \neq n(B)$ and hence A and B are not equivalent sets.

EXERCISE 1B

1. Which of the following are examples of the null set?

(i) Set of odd natural numbers divisible by 2.

(ii) Set of even prime numbers.

(iii) $A = \{x : x \in N, 1 < x \leq 2\}$

(iv) $B = \{x : x \in N, 2x + 3 = 4\}$

(v) $C = \{x : x \text{ is prime}, 90 < x < 96\}$

(vi) $D = \{x : x \in N, x^2 + 1 = 0\}$

(vii) $E = \{x : x \in W, x + 3 \leq 3\}$

(viii) $F = \{x : x \in Q, 1 < x < 2\}$

(ix) $G = \{0\}$

2. Which of the following are examples of the singleton set?

(i) $\{x : x \in Z, x^2 = 4\}$ (ii) $\{x : x \in Z, x + 5 = 0\}$

(iii) $\{x : x \in Z, |x| = 1\}$ (iv) $\{x : x \in N, x^2 = 16\}$

(v) $\{x : x$ is an even prime number$\}$

3. Which of the following are pairs of equal sets?

(i) A = set of letters in the word, 'ALLOY'
B = set of letters in the word, 'LOYAL'

(ii) C = set of letters in the word, 'CATARACT'
D = set of letters in the word 'TRACT'

(iii) $E = \{x : x \in Z, x^2 \leq 4\}$ and $F = \{x : x \in Z, x^2 = 4\}$

(iv) $G = \{-1, 1\}$ and $H = \{x : x \in Z, x^2 - 1 = 0\}$

(v) $J = \{2, 3\}$ and $K = \{x : x \in Z, (x^2 + 5x + 6) = 0\}$

4. Which of the following are pairs of equivalent sets?

(i) $A = \{-2, -1, 0\}$ and $B = \{1, 2, 3\}$

(ii) $C = \{x : x \in N, x < 3\}$ and $D = \{x : x \in W, x < 3\}$

(iii) $E = \{a, e, i, o, u\}$ and $F = \{p, q, r, s, t\}$

5. State whether the given set is finite or infinite:

(i) A = set of all triangles in a plane

(ii) B = set of all points on the circumference of a circle

(iii) C = set of all lines parallel to the y-axis

(iv) D = set of all leaves on a tree

(v) E = set of all positive integers greater than 500

(vi) $F = \{x \in R: 0 < x < 1]$

(vii) $G = \{x \in Z: x < 1\}$

(viii) $H = \{x \in Z: -15 < x < 15\}$

(ix) $J = \{x : x \in N$ and x is prime$\}$

(x) $K = \{x : x \in N$ and x is odd$\}$

(xi) L = set of all circles passing through the origin (0, 0)

6. Rewrite the following statements using set notation:

(i) a is an element of set A.

(ii) b is not an element of A.

(iii) A is an empty set and B is a nonempty set.

(iv) Number of elements in A is 6.

(v) 0 is a whole number but not a natural number.

ANSWERS (EXERCISE 1B)

1. (i), (iv), (v), (vi) **2.** (ii), (iv), (v)

3. (i) $A = B$ (ii) $C = D$ (iv) $G = H$

4. (i) A and B are equivalent sets (iii) E and F are equivalent sets

5. (i) infinite (ii) infinite (iii) infinite (iv) finite (v) infinite
(vi) infinite (vii) infinite (viii) finite (ix) infinite (x) infinite
(xi) infinite

6. (i) $a \in A$ (ii) $b \notin A$ (iii) $A = \phi$ and $B \neq \phi$ (iv) $n(A) = 6$
(v) $0 \in W$ but $0 \notin N$

SUBSETS

SUBSET *A set A is said to be a subset of set B if every element of A is also an element of B, and we write, $A \subseteq B$.*

SUPERSET *If $A \subseteq B$, then B is called a superset of A, and we write, $B \supseteq A$.*

PROPER SUBSET *If $A \subseteq B$ and $A \neq B$ then A is called a proper subset of B and we write, $A \subset B$.*

REMARK If there exists even a single element in A which is not in B, then A is not a subset of B, and we write, $A \not\subset B$.

Examples of subsets

EXAMPLE 1 Let $A = \{2, 3, 5\}$ and $B = \{2, 3, 5, 7, 9\}$.
Then, every element of A is an element of B.
$\therefore$ $A \subseteq B$ but $A \neq B$.
Hence, A is a proper subset of set B, i.e., $A \subset B$.

EXAMPLE 2 Let $A = \{1, 2\}$ and $B = \{2, 3, 5\}$.
Then, $1 \in A$ but $1 \notin B$.
$\therefore$ $A \not\subseteq B$.
Again, $3 \in B$ but $3 \notin A$.
$\therefore$ $B \not\subseteq A$.
Thus, $A \not\subseteq B$ and $B \not\subseteq A$.

EXAMPLE 3 Clearly, $N \subset W \subset Z \subset Q \subset R$.
But, $0 \in W$ and $0 \notin N$.
$\therefore$ $W \not\subseteq N$.

EXAMPLE 4 *Let* $A = \{1, \{2, 3\}, 4\}$.
Then, which of the following statements is true?
(i) $\{2, 3\} \in A$ *(ii)* $\{2, 3\} \subset A$
Rectify the wrong statement.

SOLUTION Clearly, *A* is a set containing three elements, namely 1, {2, 3} and 4.
(i) $\{2, 3\} \in A$ is a true statement.
(ii) $\{2, 3\} \subset A$ is wrong.
On rectifying this statement, we get $\{\{2, 3\}\} \subset A$ as true statement.

SOME RESULTS ON SUBSETS

THEOREM 1 *Every set is a subset of itself.*

PROOF Let *A* be any set.
Then, each element of *A* is in *A*.
$\therefore \quad A \subseteq A$.
Hence, every set is a subset of itself.

THEOREM 2 *The empty set is a subset of every set.*

PROOF Let *A* be any set and ϕ be the empty set.
Since ϕ contains no element at all, so there is no element of ϕ which is not contained in *A*.
Hence, $\phi \subset A$.

THEOREM 3 *The total number of subsets of a set containing n elements is* 2^n.

PROOF Let *A* be a finite set containing *n* elements. Then,
number of subsets of *A* each containing no element = $1 = {}^nC_0$.
number of subsets of *A* each containing 1 element = ${}^nC_1$.
number of subsets of *A* each containing 2 elements = ${}^nC_2$.
....
....
number of subsets of *A* each containing *n* elements = ${}^nC_n$.
$\therefore$ total number of subsets of *A*
$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n)$
$= (1 + 1)^n = 2^n$ [using binomial theorem].

UNIVERSAL SET *If there are some sets under consideration then there happens to be a set which is a superset of each one of the given sets. Such a set is known as the universal set for those sets. We shall denote a universal set by U.*

EXAMPLE 1 Let $A = \{1, 2, 3\}$, $b = \{2, 3, 4, 5\}$ and $C = \{6, 7\}$.
If we consider the set $U = \{1, 2, 3, 4, 5, 6, 7\}$ then clearly, *U* is a superset of each of the given sets.
Hence, *U* is the universal set.

EXAMPLE 2 When we discuss sets of lines, triangles or circles in two-dimensional geometry, the plane in which these lines, triangles or circles lie, is the universal set.

SUBSETS OF THE SET *R* OF ALL REAL NUMBERS

(i) $N = \{1, 2, 3, 4, 5, \ldots\}$ is the set of all natural numbers.

(ii) $W = \{0, 1, 2, 3, 4, 5, \ldots\}$ is the set of all whole numbers.

(iii) $Z = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$ is the set of all integers.

$Z^+ = \{1, 2, 3, 4, 5, \ldots\}$ is the set of all positive integers.

$Z^- = \{-1, -2, -3, -4, \ldots\}$ is the set of all negative integers.

Sometimes, we denote the set of all integers by *I*.

(iv) $Q = \left\{x : x = \frac{p}{q}, \text{ where, } p, q \in Z \text{ and } q \neq 0\right\}$ is the set of all rational numbers.

The set of all positive rational numbers is denoted by Q^+.

(v) $T = \{x : x \in R \text{ and } x \notin Q\}$ is the set of all irrational numbers.

INTERVALS AS SUBSETS OF *R*

Let $a, b \in R$ and $a < b$. Then, we define:

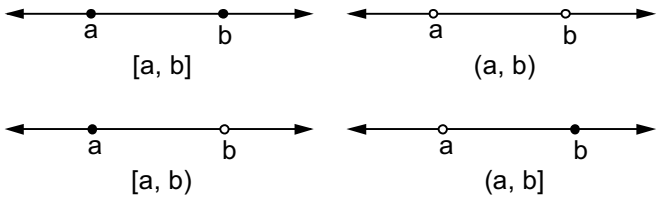
(i) *Closed Interval* $[a, b] = \{x \in R : a \leq x \leq b\}$.

(ii) *Open Interval* (a, b) or $]a, b[= \{x \in R : a < x < b\}$.

(iii) *Right Half Open Interval* $[a, b)$ or $[a, b[= \{x \in R : a \leq x < b\}$.

(iv) *Left Half Open Interval* $(a, b]$ or $]a, b] = \{x \in R : a < x \leq b\}$.

On the real line, we represent these intervals as shown below:



LENGTH OF AN INTERVAL The length of each of the intervals $[a, b]$, (a, b) $[a, b)$ and $(a, b]$ is $(b - a)$.

Examples on intervals

(i) $[-2, 3] = \{x \in R : -2 \leq x \leq 3\}$ (ii) $(-2, 3) = \{x \in R : -2 < x < 3\}$

(iii) $[-2, 3) = \{x \in R : -2 \leq x < 3\}$ (iv) $(-2, 3] = \{x \in R : -2 < x \leq 3\}$

POWER SET *The set of all subsets of a given set A is called the power set of A, denoted by* $P(A)$.

If $n(A) = m$ *then* $n[P(A)] = 2^m$.

SOLVED EXAMPLES ON SUBSETS, POWER SET AND INTERVALS

EXAMPLE 1 *Write down all possible subsets of* $A = \{4\}$.

SOLUTION All possible subsets of *A* are $\phi, \{4\}$.

$\therefore \quad P(A) = \{\phi, \{4\}\}$.

Here, $n(A) = 1$ and $n[P(A)] = 2 = 2^1$.

EXAMPLE 2 *Write down all possible subsets of* $A = \{2, 3\}$.

SOLUTION All possible subsets of *A* are

$\phi, \{2\}, \{3\}, \{2, 3\}$.

$\therefore \quad P(A) = \{\phi, \{2\}, \{3\}, \{2, 3\}\}$

Thus, $n(A) = 2$ and $n\{P(A)\} = 4 = 2^2$.

EXAMPLE 3 *Write down all possible subsets of* $A = \{-1, 0, 1\}$.

SOLUTION All possible subsets of *A* are

$\phi, \{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{0, 1\}, \{-1, 1\}$, and $\{-1, 0, 1\}$.

$\therefore \quad P(A) = \{\phi, \{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{0, 1\}, \{-1, 1\}, \{-1, 0, 1\}\}$.

Thus, $n(A) = 3$ and $n\{P(A)\} = 8 = 2^3$.

EXAMPLE 4 *Write down all possible subsets of* $A = \{1, \{2, 3\}\}$.

SOLUTION Here, *A* contains two elements, namely 1 and {2, 3}.

Let $\{2, 3\} = B$, then $A = \{1, B\}$.

$\therefore \quad P(A) = \{\phi, \{1\}, \{B\}, \{1, B\}\}$

$\Rightarrow \quad P(A) = \{\phi, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}\}$.

EXAMPLE 5 *Write down all possible subsets of* ϕ.

SOLUTION ϕ has only one subset, namely ϕ.

$\therefore \quad P(\phi) = \{\phi\}$.

EXAMPLE 6 *Write each of the following subsets of R as an interval:*

(i) $A = \{x : x \in R, -3 < x \leq 5\}$ *(ii)* $B = \{x : x \in R, -5 < x < -1\}$

(iii) $C = \{x : x \in R, -2 \leq x < 0\}$ *(iv)* $D = \{x : x \in R, -1 \leq x \leq 4\}$

Find the length of each of the above intervals.

SOLUTION We have

(i) $A = \{x : x \in R, -3 < x \leq 5\} = (-3, 5]$. Length $(A) = 5 - (-3) = 8$.

(ii) $B = \{x : x \in R, -5 < x < -1\} = (-5, -1)$. Length $(B) = -1 - (-5) = 4$.

(iii) $C = \{x : x \in R, -2 \leq x < 0\} = [-2, 0)$. Length $(C) = 0 - (-2) = 2$.

(iv) $D = \{x : x \in R, -1 \leq x \leq 4\} = [-1, 4]$. Length $(D) = 4 - (-1) = 5$.

EXAMPLE 7 *Write each of the following intervals in the set-builder form:*

(i) $A = (2, 5)$ *(ii)* $B = [-4, 7]$ *(iii)* $C = [-8, 0)$ *(iv)* $D = (5, 9]$

SOLUTION We have

(i) $A = (2, 5) = \{x : x \in R, 2 < x < 5\}$.

(ii) $B = [-4, 7] = \{x : x \in R, -4 \leq x \leq 7\}$.

(iii) $C = [-8, 0) = \{x : x \in R, -8 \leq x < 0\}$.

(iv) $D = (5, 9] = \{x : x \in R, 5 < x \leq 9\}$.

EQUAL SETS *Two sets A and B are said to be equal, if every element of A is in B and every element of B is in A, and we write, A = B.*

REMARKS (i) The elements of a set may be listed in any order.

Thus, $\{1, 2, 3\} = \{3, 1, 2\} = \{2, 3, 1\}$, etc.

(ii) The repetition of elements in a set is meaningless.

Thus, $[2, 4, 6\} = \{2, 2, 4, 6, 6\}$.

THEOREM 4 *Let A and B be two sets. Then, prove that* $A = B \Leftrightarrow A \subseteq B$ *and* $B \subseteq A$.

PROOF Let $A = B$.

Then, by definition of equal sets, every element of *A* is in *B* and every element of *B* is in *A*.

$\therefore \quad A \subseteq B$ and $B \subseteq A$.

Thus, $(A = B) \Rightarrow (A \subseteq B$ and $B \subseteq A)$.

Again, let $A \subseteq B$ and $B \subseteq A$.

Then, by the definition of a subset, it follows that every element of *A* is in *B* and every element of *B* is in *A*.

Consequently, $A = B$.

Thus, $(A \subseteq B$ and $B \subseteq A) \Rightarrow A = B$.

Hence, $A = B \Leftrightarrow (A \subseteq B$ and $B \subseteq A)$.

EXAMPLE 8 *Let* $A = \{1, \{2\}, \{3, 4\}, 5\}$. *Which of the following are incorrect statements? Rectify each:*

(i) $2 \in A$ *(ii)* $\{2\} \subset A$ *(iii)* $\{1, 2\} \subset A$

(iv) $\{3, 4\} \subset A$ *(v)* $\{1, 5\} \subset A$ *(vi)* $\{\phi\} \subset A$

(vii) $1 \subset A$ *(viii)* $\{1, 2, 3, 4\} \subset A$

SOLUTION Clearly, *A* contains four elements, namely 1, {2}, {3, 4} and 5.

(i) $2 \in A$ is incorrect. The correct statement would be $\{2\} \in A$.

(ii) $\{2\} \subset A$ is incorrect. The correct statement is $\{\{2\}\} \subset A$.

(iii) Clearly, $2 \notin A$ and therefore, $\{1, 2\} \subset A$ is incorrect.
The correct statement would be $\{1, \{2\}\} \subset A$.

(iv) Clearly, {3, 4} is an element of *A*.
So, $\{3, 4\} \subset A$ is incorrect and $\{\{3, 4\}\} \subset A$ is correct.

(v) Since 1 and 5 are both elements of *A*, so $\{1, 5\} \subset A$ is correct.

(vi) Since $\phi \notin A$, so $\{\phi\} \subset A$ is incorrect while $\phi \subset A$ is correct.

(vii) Since $1 \in A$, so $1 \subset A$ is incorrect and therefore, $\{1\} \subset A$ is correct.

(viii) Since $2 \notin A$ and $3 \notin A$, so $\{1, 2, 3, 4\} \subset A$ is incorrect.
The correct statement would be $\{1, \{2\}, \{3, 4\}\} \subset A$.

EXERCISE 1C

1. State in each case whether $A \subset B$ or $A \not\subset B$.

(i) $A = \{0, 1, 2, 3\}, B = \{1, 2, 3, 4, 5\}$

(ii) $A = \phi, B = \{0\}$

(iii) $A = \{1, 2, 3\}, B = \{1, 2, 4\}$

(iv) $A = \{x : x \in Z, x^2 = 1\}, B = \{x : x \in N, x^2 = 1\}$

(v) $A = \{x : x$ is an even natural number$\}, B = \{x : x$ is an integer$\}$

(vi) $A = \{x : x$ is an integer$\}, B = \{x : x$ is a rational number$\}$

(vii) $A = \{x : x$ is a real number$\}$, $B = \{x : x$ is a complex number$\}$

(viii) $A = \{x : x$ is an isosceles triangle in a plane$\}$,
$B = \{x : x$ is an equilateral triangle in the same plane$\}$

(ix) $A = \{x : x$ is a square in a plane$\}$,
$B = \{x : x$ is a rectangle in the same plane$\}$

(x) $A = \{x : x$ is a triangle in a plane$\}$,
$B = \{x : x$ is a rectangle in the same plane$\}$

(xi) $A = \{x : x$ is an even natural number less than 8$\}$,
$B = \{x : x$ is a natural number which divides 32$\}$

2. Examine whether the following statements are true or false:

(i) $\{a, b\} \not\subset \{b, c, a\}$ (ii) $\{a\} \in \{a, b, c\}$

(iii) $\{\phi\} \subset \{a, b, c\}$

(iv) $\{a, e\} \subset \{x : x$ is a vowel in the English alphabet$\}$

(v) $\{x : x \in W, x + 5 = 5\} = \phi$ (vi) $a \in \{\{a\}, b\}$

(vii) $\{a\} \subset \{\{a\}, b\}$ (viii) $\{b, c\} \subset \{a, \{b, c\}\}$

(ix) $\{a, a, b, b\} = \{a, b\}$ (x) $\{a, b, a, b, a, b, \ldots\}$ is an infinite set.

(xi) If A = set of all circles of unit radius in a plane and B = set of all circles in the same plane then $A \subset B$.

3. If $A = \{1\}$ and $B = \{\{1\}, 2\}$ then show that $A \not\subset B$.

Hint $1 \in A$ but $1 \notin B$.

4. Write down all subsets of each of the following sets:

(i) $A = \{a\}$ (ii) $B = \{a, b\}$ (iii) $C = \{-2, 3\}$

(iv) $D = \{-1, 0, 1\}$ (v) $E = \phi$ (vi) $F = \{2, \{3\}\}$

(vii) $G = \{3, 4, \{5, 6\}\}$

5. Express each of the following sets as an interval:

(i) $A = \{x : x \in R, -4 < x < 0\}$ (ii) $B = \{x : x \in R, 0 \le x < 3\}$

(iii) $C = \{x : x \in R, 2 < x \le 6\}$ (iv) $D = \{x : x \in R, -5 \le x \le 2\}$

(v) $E = \{x : x \in R, -3 \le x < 2\}$ (vi) $F = \{x : x \in R, -2 \le x < 0\}$

6. Write each of the following intervals in the set-builder form:

(i) $A = (-2, 3)$ (ii) $B = [4, 10]$ (iii) $C = [-1, 8)$

(iv) $D = (4, 9]$ (v) $E = [-10, 0)$ (vi) $F = (0, 5]$

7. If $A = \{3, \{4, 5\}, 6\}$, find which of the following statements are true.

(i) $\{4, 5\} \subseteq A$ (ii) $\{4, 5\} \in A$ (iii) $\{\{4, 5\}\} \subseteq A$

(iv) $4 \in A$ (v) $\{3\} \subseteq A$ (vi) $\{\phi\} \subseteq A$

(vii) $\phi \subseteq A$ (viii) $\{3, 4, 5\} \subseteq A$ (ix) $\{3, 6\} \subseteq A$

8. If $A = \{a, b, c\}$, find $P(A)$ and $n\{P(A)\}$.

9. If $A = \{1, \{2, 3\}\}$, find $P(A)$ and $n\{P(A)\}$.

10. If $A = \phi$ then find $n\{P(A)\}$.

11. If $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$ and $C = \{0, 2, 4, 6, 8\}$ then find the universal set.

12. Prove that $A \subseteq B$, $B \subseteq C$ and $C \subseteq A \Rightarrow A = C$.

13. For any set A, prove that $A \subseteq \phi \Leftrightarrow A = \phi$.

14. State whether the given statement is true or false:

(i) If $A \subset B$ and $x \notin B$ then $x \notin A$.

(ii) If $A \subseteq \phi$ then $A = \phi$.

(iii) If A, B and C are three sets such that $A \in B$ and $B \subset C$ then $A \subset C$.

Hint Let $A = \{a\}$, $B = \{\{a\}, b\}$ and $C = \{\{a\}, b, c\}$.
Then, $\{a\} \in B$ and $B \subset C$. But, $\{a\} \notin C$.

(iv) If A, B and C are three sets such that $A \subset B$ and $B \in C$ then $A \in C$.

Hint Let $A = \{a\}$, $B = \{a, b\}$ and $C = \{\{a, b\}, c\}$.
Then, $A \subset B$ and $B \in C$. But, $A \notin C$.

(v) If A, B and C are three sets such that $A \not\subset B$ and $B \not\subset C$ then $A \not\subset C$.

Hint Let $A = \{a\}$, $B = \{b, c\}$ and $C = \{a, c\}$.
Then, $A \not\subset B$ and $B \not\subset C$. But $A \subset C$.

(vi) If A and B are sets such that $x \in A$ and $A \in B$ then $x \in B$.

Hint Let $A - \{x\}$, $B - \{\{x\}, y\}$.
Then, $x \in A$ and $A \in B$. But, $x \notin B$.

ANSWERS (EXERCISE 1C)

1. (i) $A \not\subset B$ (ii) $A \subset B$ (iii) $A \not\subset B$ (iv) $A \not\subset B$ (v) $A \subset B$
(vi) $A \subset B$ (vii) $A \subset B$ (viii) $A \not\subset B$ (ix) $A \subset B$ (x) $A \not\subset B$
(xi) $A \not\subset B$

2. (i) False (ii) False (iii) False (iv) True (v) False
(vi) False (vii) False (viii) False (ix) True (x) False
(xi) True

4. (i) $\phi, \{a\}$ (ii) $\phi, \{a\}, \{b\}, \{a, b\}$ (iii) $\phi, \{-2\}, \{3\}, \{-2, 3\}$
(iv) $\phi, \{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{0, 1\}, \{-1, 1\}, \{-1, 0, 1\}$
(v) ϕ (vi) $\phi, \{2\}, \{\{3\}\}, \{2, \{3\}\}$
(vii) $\phi, \{3\}, \{4\}, \{\{5, 6\}\}, \{3, 4\}, \{3, \{5, 6\}\}, \{4, \{5, 6\}\}, \{3, 4, \{5, 6\}\}$

5. (i) $A = (-4, 0)$ (ii) $B = [0, 3)$ (iii) $C = (2, 6]$
(iv) $D = [-5, 2]$ (v) $E = [-3, 2)$ (vi) $F = [-2, 0)$

6. (i) $A = \{x : x \in R, -2 < x < 3\}$ (ii) $B = \{x : x \in R, 4 \le x \le 10\}$
(iii) $C = \{x : x \in R, -1 \le x < 8\}$ (iv) $D = \{x : x \in R, 4 < x \le 9\}$
(v) $E = \{x : x \in R, -10 \le x < 0\}$ (vi) $F = \{x : x \in R, 0 < x \le 5\}$

7. (i) False (ii) True (iii) True (iv) False (v) True
(vi) False (vii) True (viii) False (ix) True

8. $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ and $n\{P(A)\} = 2^3 = 8$

9. $P(A) = \{\phi, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}\}$ and $n\{P(A)\} = 2^2 = 4$

10. $P(A) = \{\phi\} \Rightarrow n\{P(A)\} = 1$ **11.** $U = \{0, 1, 2, 3, 4, 5, 6, 8\}$

14. (i) True (ii) True (iii) False (iv) False (v) False
(vi) False

1. (iv) $A = \{-1, 1\}$ and $B = \{1\}$. So, $A \not\subset B$.

(v) $A = \{2, 4, 6, 8, \ldots\}$ and $B = \{\ldots, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$. So, $A \subset B$.

(vi) Every integer is a rational number.

(vii) We may write a real number x as $x + 0i$.
So, every real number is a complex number.

(ix) Every square is a rectangle.

(xi) $A = \{2, 4, 6\}$ and $B = \{1, 2, 4, 8, 16, 32\}$. So, $A \not\subset B$.

2. (ii) $a \in \{a, b, c\}$ is true. So, $\{a\} \in \{a, b, c\}$ is false.

(iii) $\phi \subset \{a, b, c\}$ is true and $\{\phi\} \subset \{a, b, c\}$ is false.

(iv) $\{a, e\} \subset \{a, e, i, o, u\}$ is true.

(v) $\{x : x \in W, x + 5 = 5\} = \{0\}$.

(vi) $\{\{a\}, b\}$ has two elements, namely $\{a\}$ and b.

(vii) $\{a\} \in \{\{a\}, b\}$. So, $\{a\} \subset \{\{a\}, b\}$ is false.

(viii) $\{b, c\} \in \{a, \{b, c\}\}$. So, $\{b, c\} \subset \{a, \{b, c\}\}$ is false.

(ix) Repetition of elements in a set has no meaning.

(x) Given set $= \{a, b\}$, which is finite.

9. Let $A = \{1, a\}$. Then, $P(A) = \{\phi, \{1\}, \{a\}, \{1, a\}\}$, where $a = \{2, 3\}$.

$\therefore \quad P(A) = \{\phi, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}\}$.

OPERATIONS ON SETS

UNION OF SETS *The union of two sets A and B, denoted by $A \cup B$, is the set of all those elements which are either in A or in B or in both A and B.*

Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

$\therefore \quad x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B.$

EXAMPLE 1 *If $A = \{3, 4, 5, 6\}$ and $B = \{4, 6, 8, 10\}$, find $A \cup B$.*

SOLUTION Clearly, $A \cup B = \{3, 4, 5, 6, 8, 10\}$.

EXAMPLE 2 *Let $A = \{x : x$ is a prime number less than 10$\}$ and $B = \{x : x \in N, x$ is a factor of 12$\}$. Find $A \cup B$.*

SOLUTION We have

$A = \{2, 3, 5, 7\}$ and $B = \{1, 2, 3, 4, 6, 12\}$.

$\therefore \quad A \cup B = \{2, 3, 5, 7\} \cup \{1, 2, 3, 4, 6, 12\} = \{1, 2, 3, 4, 5, 6, 7, 12\}$.

EXAMPLE 3 *Let $A = \{x : x$ is a positive integer$\}$ and let $B = \{x : x$ is a negative integer$\}$. Find $A \cup B$.*

SOLUTION Clearly, $A \cup B = \{x : x$ is an integer and $x \neq 0\}$.

EXAMPLE 4 *If $A = \{x : x = 2n + 1, n \in Z\}$ and $B = \{x : x = 2n, n \in Z\}$ then find $A \cup B$.*

SOLUTION We have

$A \cup B = \{x : x$ is an odd integer $\} \cup \{x : x$ is an even integer$\}$

$= \{x : x$ is an integer$\} = Z$.

REMARK The union of n sets $A_1, A_2, A_3, \ldots, A_n$ is denoted by

$$(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = \bigcup_{i=1}^{n} A_i.$$

INTERSECTION OF SETS *The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements which are common to both A and B.*

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

$\therefore \quad x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B;$

$x \notin A \cap B \Rightarrow x \notin A \text{ or } x \notin B.$

EXAMPLE 5 *Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7, 11, 13\}$. Find $A \cap B$.*

SOLUTION We have

$$A \cap B = \{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 7, 11, 13\} = \{3, 5, 7\}.$$

EXAMPLE 6 *If $A = \{x : x \in N, x$ is a factor of 12$\}$ and $B = \{x : x \in N, x$ is a factor of 18$\}$, find $A \cap B$.*

SOLUTION We have

$A = \{x : x \in N, x \text{ is a factor of } 12\} = \{1, 2, 3, 4, 6, 12\},$

$B = \{x : x \in N, x \text{ is a factor of } 18\} = \{1, 2, 3, 6, 9, 18\}.$

$\therefore \quad A \cap B = \{1, 2, 3, 4, 6, 12\} \cap \{1, 2, 3, 6, 9, 18\} = \{1, 2, 3, 6\}.$

EXAMPLE 7 *If $A = \{x : x = 3n, n \in Z\}$ and $B = \{x : x = 4n, n \in Z\}$ then find $A \cap B$.*

SOLUTION We have

$A = \{x : x \in Z \text{ and } x \text{ is a multiple of } 3\}$

and $B = \{x : x \in Z \text{ and } x \text{ is a multiple of } 4\}.$

$\therefore \quad A \cap B = \{x : x \in Z \text{ and } x \text{ is a multiple of both 3 and 4}\}$

$= \{x : x \in Z \text{ and } x \text{ is a multiple of } 12\}$

$= \{x : x = 12n, n \in Z\}.$

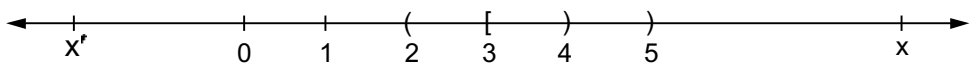
Hence, $A \cap B = \{x : x = 12n, n \in Z\}$.

EXAMPLE 8 *If $A = (2, 4)$ and $B = [3, 5)$, find $A \cap B$.*

SOLUTION We have

$A = (2, 4) = \{x : x \in R, 2 < x < 4\}$

$B = [3, 5) = \{x : x \in R, 3 \le x < 5\}$



Clearly, $A \cap B = \{x : x \in R, 3 \le x < 4\} = [3, 4)$.

REMARK The intersection of n sets $A_1, A_2, A_3, \ldots, A_n$ is denoted by

$$(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = \bigcap_{i=1}^{n} A_i.$$

DISJOINT SETS *Two sets A and B are said to be disjoint if $A \cap B = \phi$.*

INTERSECTING SETS *Two sets A and B are said to be intersecting if $A \cap B \neq \phi$.*

EXAMPLE 9 *If* $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8\}$ *and* $C = \{2, 3, 5, 7, 11\}$, *find* $(A \cap B)$ *and* $(A \cap C)$. *What do you conclude?*

SOLUTION We have

$$A \cap B = \{1, 3, 5, 7, 9\} \cap \{2, 4, 6, 8\} = \phi$$

and $A \cap C = \{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 7, 11\} = \{3, 5, 7\} \neq \phi$.

Thus, A and B are disjoint sets while A and C are intersecting sets.

EXAMPLE 10 *Give examples of three sets* A, B, C *such that* $(A \cap B) \neq \phi$, $(B \cap C) \neq \phi$, $(A \cap C) \neq \phi$ *and* $(A \cap B \cap C) = \phi$.

SOLUTION Consider the sets $A = \{1, 2\}$, $B = \{2, 3, 4\}$ and $C = \{1, 3, 5\}$.

Then, $(A \cap B) = \{1, 2\} \cap \{2, 3, 4\} = \{2\} \neq \phi$;

$(B \cap C) = \{2, 3, 4\} \cap \{1, 3, 5\} = \{3\} \neq \phi$;

$(A \cap C) = \{1, 2\} \cap \{1, 3, 5\} = \{1\} \neq \phi$;

$(A \cap B \cap C) = \{1, 2\} \cap \{2, 3, 4\} \cap \{1, 3, 5\} = \phi$.

Thus, $A \cap B \neq \phi$, $B \cap C \neq \phi$; $A \cap C \neq \phi$ and $A \cap B \cap C = \phi$.

EXAMPLE 11 *Give an example of three sets* A, B, C *such that* $A \cap B = A \cap C$ *but* $B \neq C$.

SOLUTION Consider the sets $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{3, 5, 7\}$.

Then, $A \cap B = \{1, 2, 3\} \cap \{3, 4\} = \{3\}$.

And, $A \cap C = \{1, 2, 3\} \cap \{3, 5, 7\} = \{3\}$.

Thus, $A \cap B = A \cap C$, and clearly, $B \neq C$.

DIFFERENCE OF SETS *For any sets* A *and* B, *their difference* $(A - B)$ *is defined as*

$$(A - B) = \{x : x \in A \text{ and } x \notin B\}.$$

Thus, $x \in (A - B) \Rightarrow x \in A$ *and* $x \notin B$.

EXAMPLE 12 *If* $A = \{x : x \in N, x$ *is a factor of* $6\}$ *and* $B = \{x \in N : x$ *is a factor of* $8\}$ *then find (i)* $A \cup B$, *(ii)* $A \cap B$, *(iii)* $A - B$, *(iv)* $B - A$.

SOLUTION We have

$$A = \{x : x \in N, x \text{ is a factor of } 6\} = \{1, 2, 3, 6\}$$

and $B = \{x : x \in N, x \text{ is a factor of } 8\} = \{1, 2, 4, 8\}$.

(i) $A \cup B = \{1, 2, 3, 6\} \cup \{1, 2, 4, 8\} = \{1, 2, 3, 4, 6, 8\}$.

(ii) $A \cap B = \{1, 2, 3, 6\} \cap \{1, 2, 4, 8\} = \{1, 2\}$.

(iii) $A - B = \{1, 2, 3, 6\} - \{1, 2, 4, 8\} = \{3, 6\}$.

(iv) $B - A = \{1, 2, 4, 8\} - \{1, 2, 3, 6\} = \{4, 8\}$.

SYMMETRIC DIFFERENCE OF TWO SETS

The symmetric difference of two sets A *and* B, *denoted by* $A \Delta B$, *is defined as*

$$A \Delta B = (A - B) \cup (B - A).$$

EXAMPLE 13 *Let* $A = \{a, b, c, d\}$ *and* $B = \{b, d, f, g\}$. *Find* $A \Delta B$.

SOLUTION We have

$$(A - B) = \{a, b, c, d\} - \{b, d, f, g\} = \{a, c\}.$$

$$(B - A) = \{b, d, f, g\} - \{a, b, c, d\} = \{f, g\}.$$

$$\therefore \quad A \Delta B = (A - B) \cup (B - A) = \{a, c\} \cup \{f, g\} = \{a, c, f, g\}.$$

COMPLEMENT OF A SET

Let U be the universal set and let $A \subset U$. Then, the complement of A, denoted by A' or $(U - A)$, is defined as

$$A' = \{x \in U : x \notin A\}.$$

Clearly, $x \in A' \Leftrightarrow x \notin A$.

EXAMPLE 14 *If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{2, 4, 6, 8\}$, find (i) A' (ii) $(A')'$.*

SOLUTION We have

(i) $A' = U - A$

$= \{1, 2, 3, 4, 5, 6, 7, 8\} - \{2, 4, 6, 8\} = \{1, 3, 5, 7\}$.

(ii) $(A')' = U - A'$

$= \{1, 2, 3, 4, 5, 6, 7, 8\} - \{1, 3, 5, 7\} = \{2, 4, 6, 8\} = A$.

EXAMPLE 15 *Let N be the universal set.*

(i) If $A = \{x : x \in N$ and x is odd$\}$, find A'.

(ii) If $B = \{x : x \in N, x$ is divisible by 3 and 5$\}$, find B'.

SOLUTION We have

(i) $A' = \{x : x \in N$ and x is not odd$\} = \{x : x \in N$ and x is even$\}$.

(ii) $B' = \{x : x \in N, x$ is not divisible by 3 or x is not divisible by 5$\}$.

SOME RESULTS ON COMPLEMENTATION

EXAMPLE 16 *If $A \subset U$, prove that:*

(i) $U' = \phi$ (ii) $\phi' = U$ (iii) $(A')' = A$ (iv) $A \cup A' = U$ (v) $A \cap A' = \phi$

SOLUTION We have

(i) $U' = \{x \in U : x \notin U\} = \phi$.

(ii) $\phi' = \{x \in U : x \notin \phi\} = U$.

(iii) $(A')' = \{x \in U : x \notin A'\} = [x \in U : x \in A\} = A$.

(iv) $A \cup A' = \{x \in U : x \in A \cup A'\}$

$= \{x \in U : x \in A$ or $x \in A'\}$

$= \{x \in U : x \in A$ or $x \notin A\} = U$.

(v) $A \cap A' = \{x \in U : x \in A \cap A'\}$

$= \{x \in U : x \in A$ and $x \in A'\}$

$= \{x \in U : x \in A$ and $x \notin A\} = \phi$.

EXERCISE 1D

1. If $A = \{a, b, c, d, e, f\}$, $B = \{c, e, g, h\}$ and $C = \{a, e, m, n\}$, find:

(i) $A \cup B$ (ii) $B \cup C$ (iii) $A \cup C$

(iv) $B \cap C$ (v) $C \cap A$ (vi) $A \cap B$

2. If $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, $C = \{7, 8, 9, 10, 11\}$ and $D = \{10, 11, 12, 13, 14\}$, find:

(i) $A \cup B$ (ii) $B \cup C$ (iii) $A \cup C$

(iv) $B \cup D$ (v) $(A \cup B) \cup C$ (vi) $(A \cup B) \cap C$

(vii) $(A \cap B) \cup D$ (viii) $(A \cap B) \cup (B \cap C)$ (ix) $(A \cup C) \cap (C \cup D)$

3. If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, find:

(i) $A \cap B$ (ii) $A \cap C$ (iii) $B \cap C$

(iv) $B \cap D$ (v) $B \cap (C \cup D)$ (vi) $A \cap (B \cup C)$

4. If $A = \{x : x \in N\}$, $B = \{x : x \in N$ and x is even$\}$, $C = \{x : x \in N$ and x is odd$\}$ and $D = \{x : x \in N$ and x is prime$\}$ then find:

(i) $A \cap B$ (ii) $A \cap C$ (iii) $A \cap D$

(iv) $B \cap C$ (v) $B \cap D$ (vi) $C \cap D$

5. If $A = \{2x : x \in N$ and $1 \le x < 4\}$, $B = \{(x + 2) : x \in N$ and $2 \le x < 5\}$ and $C = \{x : x \in N$ and $4 < x < 8\}$, find:

(i) $A \cap B$ (ii) $A \cup B$ (iii) $(A \cup B) \cap C$

6. If $A = \{2, 4, 6, 8, 10, 12\}$ and $B = \{3, 4, 5, 6, 7, 8, 10\}$, find:

(i) $(A - B)$ (ii) $(B - A)$ (iii) $(A - B) \cup (B - A)$

7. If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, find:

(i) $A \cap (B - C)$ (ii) $A - (B \cup C)$ (iii) $A - (B \cap C)$

8. If $A = \left\{\frac{1}{x} : x \in N \text{ and } x < 8\right\}$ *and* $B = \left\{\frac{1}{2x} : x \in N \text{ and } x \le 4\right\}$, find:

(i) $A \cup B$ (ii) $A \cap B$ (iii) $A - B$ (iv) $B - A$

9. If R is the set of all real numbers and Q is the set of all rational numbers then what is the set $(R - Q)$?

10. If $A = \{2, 3, 5, 7, 11\}$ and $B = \phi$, find:

(i) $A \cup B$ (ii) $A \cap B$

11. If A and B are two sets such that $A \subseteq B$ then find:

(i) $A \cup B$ (ii) $A \cap B$ (iii) $A - B$

12. Which of the following sets are pairs of disjoint sets? Justify your answer:

(i) $A = \{3, 4, 5, 6\}$ and $B = \{2, 5, 7, 9\}$

(ii) $C = \{1, 2, 3, 4, 5\}$ and $D = \{6, 7, 9, 11\}$

(iii) $E = \{x : x \in N, x$ is even and $x < 8\}$

$F = \{x : x = 3n, n \in N$ and $n < 4\}$

(iv) $G = \{x : x \in N, x$ is even$\}$ and $H = \{x : x \in N, x$ is prime$\}$

(v) $J = \{x : x \in N, x$ is even$\}$ and $K = \{x : x \in N, x$ is odd$\}$

13. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{1, 4, 5, 6\}$, find:

(i) A' (ii) B' (iii) C' (iv) $(B')'$

(v) $(A \cup B)'$ (vi) $(A \cap C)'$ (vii) $(B - C)'$

14. If $U = \{a, b, c, d, e\}$, $A = \{a, b, c\}$ *and* $B = \{b, c, d, e\}$ then verify that:

(i) $(A \cup B)' = (A' \cap B')$ (ii) $(A \cap B)' = (A' \cup B')$

15. If U is the universal set and $A \subset U$ then fill in the blanks:

(i) $A \cup A' =$ ..... (ii) $A \cap A' =$ .....

(iii) $\phi' \cap A =$ ..... (iv) $U' \cap A =$ .....

ANSWERS (EXERCISE 1D)

1. (i) $\{a, b, c, d, e, f, g, h\}$ (ii) $\{a, c, e, g, h, m, n\}$ (iii) $\{a, b, c, d, e, f, m, n\}$
(iv) $\{e\}$ (v) $\{a, e\}$ (vi) $\{c, e\}$

2. (i) $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (ii) $\{4, 5, 6, 7, 8, 9, 10, 11\}$
(iii) $\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11\}$ (iv) $\{4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$
(v) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ (vi) $\{7, 8\}$
(vii) $\{4, 5, 10, 11, 12, 13, 14\}$ (viii) $\{4, 5, 7, 8\}$
(ix) $\{7, 8, 9, 10, 11\}$

3. (i) $\{7, 9\}$ (ii) $\{11\}$ (iii) $\{11, 13\}$ (iv) ϕ
(v) $\{11, 13\}$ (vi) $\{7, 9, 11\}$

4. (i) B (ii) C (iii) D (iv) ϕ
(v) $\{2\}$ (vi) $D - \{2\}$

5. (i) $\{4, 6\}$ (ii) $\{2, 4, 5, 6\}$ (iii) $\{5, 6\}$

6. (i) $\{2, 12\}$ (ii) $\{3, 5, 7\}$ (iii) $\{2, 3, 5, 7, 12\}$

7. (i) $\{a, c\}$ (ii) $\{d\}$ (iii) $\{a, b, c, d\}$

8. (i) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\right\}$ (ii) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}\right\}$ (iii) $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\right\}$ (iv) $\left\{\frac{1}{8}\right\}$

9. $(R - Q) = \{x : x \in R, x \text{ is irrational}\}$

10. (i) 2, 3, 5, 7, 11} (ii) ϕ **11.** (i) B (ii) A (iii) ϕ

12. (ii) C and D, since $C \cap D = \phi$ (v) J and K, since $J \cap K = \phi$

13. (i) $\{5, 6, 7, 8, 9\}$ (ii) $\{1, 3, 5, 7, 9\}$ (iii) $\{2, 3, 7, 8, 9\}$ (iv) $\{2, 4, 6, 8\}$
(v) $\{5, 7, 9\}$ (vi) $\{2, 3, 5, 6, 7, 8, 9\}$ (vii) $\{1, 3, 4, 5, 6, 7, 9\}$

15. (i) U (ii) ϕ (iii) A (iv) ϕ

LAWS OF OPERATIONS ON SETS

THEOREM 1 (Idempotent laws) *For any set A, prove that:*

(i) $A \cup A = A$ *(ii)* $A \cap A = A$

PROOF We have

(i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A.$

(ii) $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A.$

THEOREM 2 (Identity laws) *For any set A, prove that:*

(i) $A \cup \phi = A$ *(ii)* $A \cap U = A$, *where U is the universal set*

PROOF We have

(i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$ $[\because\ x \notin \phi]$

(ii) $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$ $[\because\ A \subset U]$

NOTE ϕ and U are the identity elements for union and intersection of sets respectively.

THEOREM 3 (Commutative laws) *For any two sets A and B, prove that:*

I. $A \cup B = B \cup A$ [*Commutative law for union of sets*]

II. $A \cap B = B \cap A$ [*Commutative law for intersection of sets*]

PROOF I. Let x be an arbitrary element of $A \cup B$. Then,

$$x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$$
$$\Rightarrow x \in B \text{ or } x \in A$$
$$\Rightarrow x \in B \cup A.$$

$\therefore \quad A \cup B \subseteq B \cup A.$ ... (i)

Again, let y be an arbitrary element of $B \cup A$. Then,

$$y \in B \cup A \Rightarrow y \in B \text{ or } y \in A$$
$$\Rightarrow y \in A \text{ or } y \in B$$
$$\Rightarrow y \in A \cup B.$$

$\therefore \quad B \cup A \subseteq A \cup B.$ ... (ii)

From (i) and (ii), we get $A \cup B = B \cup A$.

II. Let x be an arbitrary element of $A \cap B$. Then,

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$
$$\Rightarrow x \in B \text{ and } x \in A$$
$$\Rightarrow x \in B \cap A.$$

$\therefore \quad A \cap B \subseteq B \cap A.$ ... (iii)

Again, let y be an arbitray element of $B \cap A$. Then,

$$y \in B \cap A \Rightarrow y \in B \text{ and } y \in A$$
$$\Rightarrow y \in A \text{ and } y \in B$$
$$\Rightarrow y \in A \cap B.$$

$\therefore \quad B \cap A \subseteq A \cap B.$ ... (iv)

From (iii) and (iv), we get $A \cap B = B \cap A$.

THEOREM 4 (Associative laws) *For any sets A, B, C, prove that:*

I. $(A \cup B) \cup C = A \cup (B \cup C)$ [*Associative law for union of sets*]

II. $(A \cap B) \cap C = A \cap (B \cap C)$ [*Associative law for intersection of sets*]

PROOF I. Let x be an arbitrary element of $(A \cup B) \cup C$. Then,

$$x \in (A \cup B) \cup C \Rightarrow x \in (A \cup B) \text{ or } x \in C$$
$$\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C$$
$$\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C)$$
$$\Rightarrow x \in A \text{ or } x \in (B \cup C)$$
$$\Rightarrow x \in A \cup (B \cup C).$$

$\therefore \quad (A \cup B) \cup C \subseteq A \cup (B \cup C)$. ... (i)

Again, let y be an arbitrary element of $A \cup (B \cup C)$. Then,

$$\begin{aligned} y \in A \cup (B \cup C) &\Rightarrow y \in A \text{ or } y \in (B \cup C) \\ &\Rightarrow y \in A \text{ or } (y \in B \text{ or } y \in C) \\ &\Rightarrow (y \in A \text{ or } y \in B) \text{ or } y \in C \\ &\Rightarrow y \in (A \cup B) \text{ or } y \in C \\ &\Rightarrow y \in (A \cup B) \cup C. \end{aligned}$$

$\therefore \quad A \cup (B \cup C) \subseteq (A \cup B) \cup C$. ... (ii)

From (i) and (ii), we get $(A \cup B) \cup C = A \cup (B \cup C)$.

II. Let x be an arbitrary element of $(A \cap B) \cap C$. Then,

$$\begin{aligned} x \in (A \cap B) \cap C &\Rightarrow x \in (A \cap B) \text{ and } x \in C \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C \\ &\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \in C) \\ &\Rightarrow x \in A \text{ and } x \in (B \cap C) \\ &\Rightarrow x \in A \cap (B \cap C). \end{aligned}$$

$\therefore \quad (A \cap B) \cap C \subseteq A \cap (B \cap C)$. ... (iii)

Again, let y be an arbitrary element of $A \cap (B \cap C)$. Then,

$$\begin{aligned} y \in A \cap (B \cap C) &\Rightarrow y \in A \text{ and } y \in (B \cap C) \\ &\Rightarrow y \in A \text{ and } (y \in B \text{ and } y \in C) \\ &\Rightarrow (y \in A \text{ and } y \in B) \text{ and } y \in C \\ &\Rightarrow y \in (A \cap B) \text{ and } y \in C \\ &\Rightarrow y \in (A \cap B) \cap C \end{aligned}$$

$\therefore \quad A \cap (B \cap C) \subseteq (A \cap B) \cap C$. ... (iv)

From (iii) and (iv), we get $(A \cap B) \cap C = A \cap (B \cap C)$.

THEOREM 5 (Distributive laws) *For any three sets A, B, C prove that:*

I. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

[*Distributive law of union over intersection*]

II. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

[*Distributive law of intersection over union*]

PROOF I. Let x be an arbitrary element of $A \cup (B \cap C)$. Then,

$$\begin{aligned} x \in A \cup (B \cap C) &\Rightarrow x \in A \text{ or } x \in (B \cap C) \\ &\Rightarrow x \in A \text{ or } (x \in B) \text{ and } x \in C \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)) \\ &\qquad [\because \text{ 'or' distributes 'and'}] \\ &\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C). \\ &\Rightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

$\therefore \quad A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. ... (i)

Again, let y be an arbitrary element of $(A \cup B) \cap (A \cup C)$. Then,

$y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow (y \in A$ or $y \in B)$ and $(y \in A$ or $y \in C)$

$\Rightarrow y \in A$ or $(y \in B$ and $y \in C)$

[$\because$ 'or' distributes 'and']

$\Rightarrow y \in A$ or $y \in (B \cap C)$

$\Rightarrow y \in A \cup (B \cap C)$.

$\therefore \quad (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. … (ii)

From (i) and (ii), we get $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

II. Let x be an arbitrary element of $A \cap (B \cup C)$. Then,

$x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

[$\because$ 'and' distributes 'or']

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$.

$\therefore \quad A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. … (iii)

Again, let y be an arbitrary element of $(A \cap B) \cup (A \cap C)$. Then,

$y \in (A \cap B) \cup (A \cap C) \Rightarrow y \in (A \cap B)$ or $y \in (A \cap C)$

$\Rightarrow (y \in A$ and $y \in B)$ or $(y \in A$ and $y \in C)$

$\Rightarrow y \in A$ and $(y \in B$ or $y \in C)$

[$\because$ 'and' distributes 'or']

$\Rightarrow y \in A$ and $y \in (B \cup C)$

$\Rightarrow y \in A \cap (B \cup C)$.

$\therefore \quad (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. … (iv)

From (iii) and (iv), we get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

THEOREM 6 (De Morgan's laws) *For any two sets A and B, prove that:*

I. $(A \cup B)' = (A' \cap B')$ *II.* $(A \cap B)' = (A' \cup B')$

PROOF I. Let x be an arbitrary element of $(A \cup B)'$. Then,

$x \in (A \cup B)' \Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$ [*note this point*]

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in (A' \cap B')$.

$\therefore \quad (A \cup B)' \subseteq (A' \cap B')$. … (i)

Again, let y be an arbitrary element of $(A' \cap B')$. Then,

$y \in (A' \cap B') \Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$ [*note this point*]

$\Rightarrow y \in (A \cup B)'$.

$\therefore \quad (A' \cap B') \subseteq (A \cup B)'$.

From (i) and (ii), we get $(A \cup B)' = (A' \cap B')$. ... (ii)

II. Let x be an arbitrary element of $(A \cap B)'$. Then,

$$x \in (A \cap B)' \Rightarrow x \notin (A \cap B)$$
$$\Rightarrow x \notin A \text{ or } x \notin B \quad [\textit{note this point}]$$
$$\Rightarrow x \in A' \text{ or } x \in B'$$
$$\Rightarrow x \in (A' \cup B')$$

$\therefore \quad (A \cap B)' \subseteq (A' \cup B')$. ... (iii)

Again, let y be an arbitrary element of $(A' \cup B')$. Then,

$$y \in (A' \cup B') \Rightarrow y \in A' \text{ or } y \in B'$$
$$\Rightarrow y \notin A \text{ or } y \notin B$$
$$\Rightarrow y \notin (A \cap B) \quad [\textit{note this point}]$$
$$\Rightarrow y \in (A \cap B)'.$$

$\therefore \quad (A' \cup B') \subseteq (A \cap B)'$. ... (iv)

From (iii) and (iv), we get $(A \cap B)' = (A' \cup B')$.

SOME MORE RESULTS

THEOREM 7 *For any two sets A and B, prove that:*

I. $A \subseteq B \Rightarrow B' \subseteq A'$ *II.* $A - B = A \cap B'$

III. $(A - B) \cap B = \phi$ *IV.* $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

V. $(A - B) = A \Leftrightarrow A \cap B = \phi$

PROOF I. Let $A \subseteq B$ be given and let x be an arbitrary element of B'. Then,

$$x \in B' \Rightarrow x \notin B$$
$$\Rightarrow x \notin A \quad [\because A \subseteq B]$$
$$\Rightarrow x \in A'.$$

$\therefore \quad B' \subseteq A'$.

Hence, $A \subseteq B \Rightarrow B' \subseteq A'$.

II. Let x be an arbitrary element of $(A - B)$. Then,

$$x \in (A - B) \Rightarrow x \in A \text{ and } x \notin B$$
$$\Rightarrow x \in A \text{ and } x \in B'$$
$$\Rightarrow x \in A \cap B'.$$

$\therefore \quad (A - B) \subseteq A \cap B'$. ... (i)

Again, let y be an arbitrary element of $(A \cap B')$. Then,

$$x \in (A \cap B') \Rightarrow x \in A \text{ and } x \in B'$$
$$\Rightarrow x \in A \text{ and } x \notin B$$
$$\Rightarrow x \in (A - B).$$

$\therefore \quad (A \cap B') \subseteq (A - B)$. ... (ii)

Hence, from (i) and (ii), we get $(A - B) = (A \cap B')$.

III. If possible, let $(A - B) \cap B \neq \phi$ and let $x \in (A - B) \cap B$. Then,

$$x \in (A - B) \cap B \Rightarrow x \in (A - B) \text{ and } x \in B$$

$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \in B$

$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \in B).$

But, $x \notin B$ and $x \in B$ can never hold simultaneously.

Thus, we arrive at a contradiction.

Since the contradiction arises by assuming that $(A - B) \cap B \neq \phi$ and hence $(A - B) \cap B = \phi$.

IV. Let x be an arbitrary element of $(A - B) \cup (B - A)$. Then,

$x \in (A - B) \cup (B - A) \Rightarrow x \in (A - B) \text{ or } x \in (B - A)$

$\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)$

$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \notin A \text{ or } x \notin B)$

[*note it*]

$\Rightarrow x \in (A \cup B) \text{ and } x \notin (A \cap B)$

$\Rightarrow x \in \{(A \cup B) - (A \cap B)\}.$

$\therefore \quad (A - B) \cup (B - A) \subseteq \{(A \cup B) - (A \cap B)\}. \qquad \ldots (iii)$

Again, let y be an arbitrary element of $(A \cup B) - (A \cap B)$. Then,

$y \in (A \cup B) - (A \cap B) \Rightarrow y \in (A \cup B) \text{ and } y \notin (A \cap B)$

$\Rightarrow (y \in A \text{ or } y \in B) \text{ and } (y \notin A \text{ or } y \notin B)$

[*note it*]

$\Rightarrow (y \in A \text{ and } y \notin B) \text{ or } (y \in B \text{ and } y \notin A)$

[*note it*]

$\Rightarrow y \in (A - B) \text{ or } y \in (B - A)$

$\Rightarrow y \in (A - B) \cup (B - A).$

$\therefore \quad (A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A). \qquad \ldots (iv)$

From (iii) and (iv), we get $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

V. Let $(A - B) = A$ be given and we have to show that $A \cap B = \phi$.

If possible, let $A \cap B \neq \phi$ and let $x \in A \cap B$. Then,

$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$

$\Rightarrow x \in (A - B) \text{ and } x \in B \quad [\because A = (A - B) \text{ (given)}]$

$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \in B$

$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \in B).$

But, $x \notin B$ and $x \in B$ both can never hold simultaneously.

Thus, we arrive at a contradiction.

Since the contradiction arises by assuming that $A \cap B \neq \phi$, so $A \cap B = \phi$.

Thus, $(A - B) = A \Rightarrow A \cap B = \phi$.

Again, let $(A \cap B) = \phi$ and we have to show that $(A - B) = A$.

Now, $x \in (A - B) \Rightarrow x \in A \text{ and } x \notin B$

$\Rightarrow x \in A$ (surely).

$\therefore \quad (A - B) \subseteq A. \qquad \ldots (v)$

Again, $y \in A \Rightarrow y \notin B$ $[\because A \cap B = \phi]$

$\Rightarrow y \in A$ and $y \notin B$

$\Rightarrow y \in (A - B)$.

$\therefore \quad A \subseteq (A - B)$. ... (vi)

Thus, from (v) and (vi), we get $(A - B) = A$.

$\therefore \quad A \cap B = \phi \Rightarrow (A - B) = A$.

Hence, $(A - B) = A \Leftrightarrow (A \cap B) = \phi$.

THEOREM 8 *If $(A \cup B) = (A \cap B)$ then prove that $A = B$.*

PROOF Let $(A \cup B) = (A \cap B)$ be given.

Let x be an arbitrary element of A. Then,

$x \in A \Rightarrow x \in A \cup B$ $[\because A \subseteq A \cup B]$

$\Rightarrow x \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$ (surely).

$\therefore \quad A \subseteq B$. ... (i)

Again, let $y \in B$. Then,

$y \in B \Rightarrow y \in A \cup B$ $[\because B \subseteq A \cup B]$

$\Rightarrow y \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow y \in A$ and $y \in B$

$\Rightarrow y \in A$ (surely).

$\therefore \quad B \subseteq A$. ... (ii)

Thus, from (i) and (ii), we get $A = B$.

THEOREM 9 *If $A \subseteq B$ then for any set C, prove that $(C - B) \subseteq (C - A)$.*

PROOF Let $A \subseteq B$ be given.

Let $x \in (C - B)$. Then,

$x \in (C - B) \Rightarrow x \in C$ and $x \notin B$

$\Rightarrow x \in C$ and $x \notin A$ $[\because A \subseteq B]$

$\Rightarrow x \in (C - A)$.

$\therefore \quad (C - B) \subseteq (C - A)$.

Hence, $A \subseteq B \Rightarrow (C - B) \subseteq (C - A)$.

THEOREM 10 *For any sets A and B, prove that:*

(i) $A \cup (A \cap B) = A$ *(ii)* $A \cap (A \cup B) = A$

PROOF (i) Since $(A \cap B) \subseteq A$, we have $A \cup (A \cap B) = A$ $[\because X \subseteq Y \Rightarrow X \cup Y = Y]$.

(ii) Since $A \subseteq (A \cup B)$, we have $A \cap (A \cup B) = A$ $[\because X \subseteq Y \Rightarrow X \cap Y = X]$.

THEOREM 11 *For any sets A and B, prove that:*

(i) $(A \cap B) \cup (A - B) = A$ *(ii)* $A \cup (B - A) = (A \cup B)$

PROOF (i) We have

$$(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B') \quad [\because (A - B) = (A \cap B')]$$
$$= A \cap (B \cup B') \quad \text{[by distributive law]}$$
$$= A \cap U \quad [\because B \cup B' = U]$$
$$= A.$$

Hence, $(A \cap B) \cup (A - B) = A$.

(ii) We have

$$A \cup (B - A) = A \cup (B \cap A') \quad [\because (B - A) = (B \cap A')]$$
$$= (A \cup B) \cap (A \cup A') \quad \text{[by distributive law]}$$
$$= (A \cup B) \cap U \quad [\because A \cup A' = U]$$
$$= (A \cup B).$$

Hence, $A \cup (B - A) = (A \cup B)$.

THEOREM 12 *If $A \cap B' = \phi$ then prove that $A = A \cap B$ and hence show that $A \subseteq B$.*

PROOF Let $A \cap B' = \phi$ be given. Then,

$$A = (A \cap U), \text{ where } U \text{ is the universal set}$$
$$= A \cap (B \cup B') \quad [\because B \cup B' = U]$$
$$= (A \cap B) \cup (A \cap B')$$
$$= (A \cap B) \cup \phi \quad [\because A \cap B' = \phi]$$
$$= (A \cap B).$$

Hence, $A = (A \cap B)$.

Further, let $A = A \cap B$ and let $x \in A$. Then,

$$x \in A \Rightarrow x \in A \cap B \quad [\because A = A \cap B]$$
$$\Rightarrow x \in A \text{ and } x \in B$$
$$\Rightarrow x \in B \text{ (surely)}.$$

$\therefore \quad A \subseteq B.$

THEOREM 13 *If A, B and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$ then prove that $B = C$.*

PROOF Let $A \cup B = A \cup C$ and $A \cap B = A \cap C$ be given. Then,

$$A \cup B = A \cup C$$
$\Rightarrow \quad (A \cup B) \cap B = (A \cup C) \cap B$ and $(A \cup B) \cap C = (A \cup C) \cap C$
$\Rightarrow \quad B = (A \cap B) \cup (C \cap B)$ and $(A \cap C) \cup (B \cap C) = C$
$$[\because B \subseteq (A \cup B) \text{ and } C \subseteq (A \cup C)]$$
$\Rightarrow \quad B = (A \cap B) \cup (B \cap C)$ and $(A \cap B) \cup (B \cap C) = C \quad [\because A \cap C = A \cap B]$
$\Rightarrow \quad B = C.$

Hence, $B = C$.

THEOREM 14 *For any sets A, B and C, prove that:*

(i) $A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $A - (B \cap C) = (A - B) \cup (A - C)$

(iii) $(A \cup B) - C = (A - C) \cup (B - C)$

(iv) $(A \cap B) - C = (A - C) \cap (B - C)$

PROOF We have

(i) $A-(B\cup C)=A\cap(B\cup C)'$ [$\because\ X-Y=X\cap Y'$]

$=A\cap(B'\cap C')$ [by De Morgan's law]

$=(A\cap B')\cap(A\cap C')$

$=(A-B)\cap(A-C)$.

$\therefore\quad A-(B\cup C)=(A-B)\cap(A-C)$.

(ii) $A-(B\cap C)=A\cap(B\cap C)'$ [$\because\ X-Y=X\cap Y'$]

$=A\cap(B'\cup C')$ [by De Morgan's law]

$=(A\cap B')\cup(A\cap C')$ [by distributive law]

$=(A-B)\cup(A-C)$ [$\because\ X\cap Y'=X-Y$].

$\therefore\quad A-(B\cap C)=(A-B)\cup(A-C)$.

(iii) $(A\cup B)-C=(A\cup B)\cap C'$ [$\because\ X-Y=X\cap Y'$]

$=(A\cap C')\cup(B\cap C')$ [by distributive law]

$=(A-C)\cup(B-C)$. [$\because\ X\cap Y'=X-Y$]

$\therefore\quad (A\cup B)-C=(A-C)\cup(B-C)$.

(iv) $(A\cap B)-C=(A\cap B)\cap C'$ [$\because\ X-Y=X\cap Y'$]

$=(A\cap C')\cap(B\cap C')$ [*note*]

$=(A-C)\cap(B-C)$. [$\because\ X\cap Y'=X-Y$]

$\therefore\quad (A\cap B)-C=(A-C)\cap(B-C)$.

THEOREM 15 *Let A and B be sets. If* $A\cap X=B\cap X=\phi$ *and* $A\cup X=B\cup X$ *for some set X, show that* $A=B$.

PROOF $A\cup X=B\cup X$ [given]

$\Rightarrow\quad A\cap(A\cup X)=A\cap(B\cup X)$

$\Rightarrow\quad (A\cap A)\cup(A\cap X)=(A\cap B)\cup(A\cap X)$ [by distributive law]

$\Rightarrow\quad A\cup\phi=(A\cap B)\cup\phi$ [$\because\ A\cap X=\phi$]

$\Rightarrow\quad A=(A\cap B)$

$\Rightarrow\quad A\subseteq B$ ... (i) [$\because\ A\cap B=A\ \Rightarrow\ A\subseteq B$].

Again, $A\cup X=B\cup X$

$\Rightarrow\quad B\cap(A\cup X)=B\cap(B\cup X)$

$\Rightarrow\quad (B\cap A)\cup(B\cap X)=(B\cap B)\cup(B\cap X)$ [by distributive law]

$\Rightarrow\quad (A\cap B)\cup\phi=B\cup\phi$ [$\because\ B\cap X=\phi$ and $B\cap A=A\cap B$]

$\Rightarrow\quad A\cap B=B$

$\Rightarrow\quad B\subseteq A$ ... (ii) [$\because\ A\cap B=B\ \Rightarrow\ B\subseteq A$].

From (i) and (ii), we get $A=B$.

THEOREM 16 *Show that the following four conditions are equivalent:*

(i) $A\subset B$ *(ii)* $A-B=\phi$ *(iii)* $A\cup B=B$ *(iv)* $A\cap B=A$

PROOF In order to prove the required result, we will show that:

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Now, **(i) $\Rightarrow$ (ii):**

Let $A \subset B$ be given.

Then, there is no element of *A* which is not in *B*.

$\therefore \quad A - B = \{x : x \in A \text{ and } x \notin B\} = \phi$

[$\because$ there is no element of *A* which is not in *B*].

Hence, $A \subset B \Rightarrow A - B = \phi$ and therefore, (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii):

Let $A - B = \phi$ be given. Then,

$A - B = \phi \Rightarrow$ every element of *A* is in *B*

$\Rightarrow A \subset B$

$\Rightarrow A \cup B = B.$

Thus, $A - B = \phi \Rightarrow A \cup B = B$ and therefore, (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv):

Let $A \cup B = B$ be given. Then,

$A \cup B = B \Rightarrow A \subset B \Rightarrow A \cap B = A.$

Thus, $A \cup B = B \Rightarrow A \cap B = A$ and therefore, (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i):

Let $A \cap B = A$ be given. Then,

$x \in A \Rightarrow x \in A \cap B$ [$\because A = A \cap B$]

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$ (surely).

$\therefore \quad A \subseteq B.$

Thus, $A \cap B = A \Rightarrow A \subseteq B$ and therefore, (iv) $\Rightarrow$ (i).

$\therefore$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Hence, the given four conditions are equivalent.

THEOREM 17 *For any sets A and B, prove that*

$$P(A \cap B) = P(A) \cap P(B).$$

PROOF Let $X \in P(A \cap B)$. Then,

$X \in P(A \cap B) \Rightarrow X \subseteq A \cap B$

$\Rightarrow X \subseteq A$ and $X \subseteq B$

$\Rightarrow X \in P(A)$ and $X \in P(B)$

$\Rightarrow X \in P(A) \cap P(B).$

$\therefore \quad P(A \cap B) \subseteq P(A) \cap P(B).$ ... (i)

Again, let $Y \in P(A) \cap P(B)$. Then,

$Y \in P(A) \cap P(B) \Rightarrow Y \in P(A)$ and $Y \in P(B)$

$\Rightarrow Y \subseteq A$ and $Y \subseteq B$

$\Rightarrow Y \subseteq A \cap B$

$\Rightarrow Y \in P(A \cap B).$

$\therefore \quad P(A) \cap P(B) \subseteq P(A \cap B).$ ... (ii)

From (i) and (ii), we get $P(A \cap B) = P(A) \cap P(B)$.

THEOREM 18 *For any two sets A and B, prove that*

$$P(A) \cup P(B) \subset P(A \cup B).$$

But, $P(A \cup B)$ is not necessarily a subset of $P(A) \cup P(B)$.

PROOF Let X be an arbitrary element of $P(A) \cup P(B)$. Then,

$$\begin{aligned} X \in P(A) \cup P(B) &\Rightarrow X \in P(A) \text{ or } X \in P(B) \\ &\Rightarrow X \subset A \text{ or } X \subset B \\ &\Rightarrow X \subset (A \cup B) \\ &\Rightarrow X \in P(A \cup B). \end{aligned}$$

$\therefore \quad P(A) \cup P(B) \subset P(A \cup B).$

However, $P(A \cup B) \subset P(A) \cup P(B)$ is not always true.

For example, let $A = \{1\}$ and $B = \{2\}$. Then, $A \cup B = \{1, 2\}$.

$\therefore \quad P(A) = \{\phi, \{1\}\}, P(B) = \{\phi, \{2\}\}$

and $P(A \cup B) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$.

Also, $P(A) \cup P(B) = \{\phi, \{1\}, \{2\}\}$.

$\therefore \quad P(A \cup B) \not\subset P(A) \cup P(B).$

Hence, in general, $P(A \cup B) \neq P(A) \cup P(B)$.

THEOREM 19 *If $P(A) = P(B)$, prove that $A = B$.*

PROOF Let $P(A) = P(B)$. Then,

$$\begin{aligned} A \subseteq A &\Rightarrow A \in P(A) \\ &\Rightarrow A \in P(B) && [\because P(A) = P(B)] \\ &\Rightarrow A \subseteq B. && \text{... (i)} \end{aligned}$$

$$\begin{aligned} \text{Again, } B \subseteq B &\Rightarrow B \in P(B) \\ &\Rightarrow B \in P(A) && [\because P(B) = P(A)] \\ &\Rightarrow B \subseteq A. && \text{... (ii)} \end{aligned}$$

From (i) and (ii), we get $A \subseteq B$ and $B \subseteq A$.

Hence, $A = B$.

EXERCISE 1E

1. If $A = \{a, b, c, d, e\}, B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, verify that:

(i) $A \cup B = B \cup A$ (ii) $A \cup C = C \cup A$ (iii) $B \cup C = C \cup B$

(iv) $A \cap B = B \cap A$ (v) $B \cap C = C \cap B$ (vi) $A \cap C = C \cap A$

(vii) $(A \cup B) \cup C = A \cup (B \cup C)$ (viii) $(A \cap B) \cap C = A \cap (B \cap C)$

2. If $A = \{a, b, c, d, e\}, B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, verify that:

(i) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (ii) $A - (B \cap C) = (A - B) \cup (A - C)$

3. If $A = \{x : x \in N, x \leq 7\}, B = \{x : x$ is prime, $x < 8\}$ and $C = \{x : x \in N, x$ is odd and $x < 10\}$, verify that:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$, verify that:

(i) $(A \cup B)' = (A' \cap B')$ (ii) $(A \cap B)' = (A' \cup B')$

5. Let $A = \{a, b, c\}$, $B = \{b, c, d, e\}$ and $C = \{c, d, e, f\}$ be subsets of $U = \{a, b, c, d, e, f\}$. Then, verify that:

(i) $(A')' = A$ (ii) $(A \cup B)' = (A' \cap B')$ (iii) $(A \cap B)' = (A' \cup B')$

6. Given an example of three sets *A*, *B*, *C* such that $A \cap B \neq \phi$, $B \cap C \neq \phi$, $A \cap C \neq \phi$ and $A \cap B \cap C = \phi$.

7. For any sets *A* and *B*, prove that:

(i) $(A - B) \cap B = \phi$ (ii) $A \cup (B - A) = A \cup B$

(iii) $(A - B) \cup (A \cap B) = A$ (iv) $(A \cup B) - B = A - B$

(v) $A - (A \cap B) = A - B$

8. For any sets *A* and *B*, prove that:

(i) $A \cap B' = \phi \Rightarrow A \subset B$ (ii) $A' \cup B = U \Rightarrow A \subset B$

HINTS TO SOME SELECTED QUESTIONS

6. Take $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 3\}$.

7. (i) $(A - B) \cap B = (A \cap B') \cap B = A \cap (B' \cap B) = A \cap \phi = \phi$.

(ii) $A \cup (B - A) = A \cup (B \cap A') = (A \cup B) \cap (A \cup A') = (A \cup B) \cap U = (A \cup B)$.

(iii) $(A - B) \cup (A \cap B) = (A \cap B') \cup (A \cap B) = A \cap (B' \cup B) = A \cap U = A$.

(iv) $(A \cup B) - B = (A \cup B) \cap B' = (A \cap B') \cup (B \cap B') = (A \cap B') \cup \phi = A \cap B' = A - B$.

(v) $A - (A \cap B) = A \cap (A \cap B)' = A \cap (A' \cup B') = (A \cap A') \cup (A \cap B') = \phi \cup (A \cap B')$
$= (A \cap B') = (A - B)$.

8. (i) $A \cap B' = \phi \Rightarrow A - B = \phi \Rightarrow A \subset B$.

(ii) $A' \cup B = U \Rightarrow A \cap (A' \cup B) = A \cap U \Rightarrow (A \cap A') \cup (A \cap B) = A$
$\Rightarrow \phi \cup (A \cap B) = A \Rightarrow A \cap B = A \Rightarrow A \subset B$.

VENN DIAGRAMS

In order to express the relationship among sets in perspective, we represent them pictorially by means of diagrams, called *Venn diagrams*.

In these diagrams, the universal set is represented by a rectangular region and its subsets by circles inside the rectangle. We represent disjoint sets by disjoint circles and intersecting sets by intersecting circles.

VENN DIAGRAMS IN DIFFERENT SITUATIONS

CASE 1 *When the universal set and its subset are given*

Let *U* be the universal set and let $A \subseteq U$.

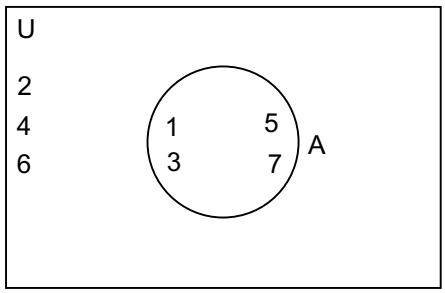
We draw a circle inside a rectangle.

The rectangular region represents *U* and the circular region represents *A*.

EXAMPLE Let $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{1, 3, 5, 7\}$.

Then, we draw the Venn diagram, as shown in the given figure.

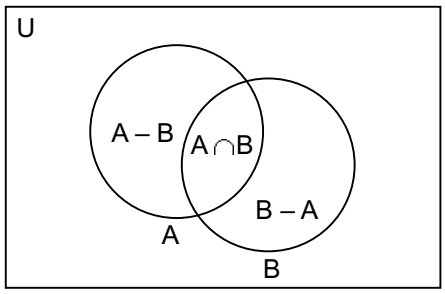
Clearly, $A' = \{2, 4, 6\}$.



CASE 2 *When two intersecting subsets of U are given*

For representing two intersecting subsets *A* and *B* of *U*, we draw two intersecting circles within the rectangle.

The common region of these circles represents $A \cap B$.



Excluding the region of *B* from that of *A* shows $(A - B)$.

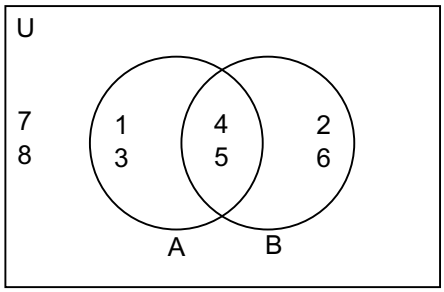
Excluding the region of *A* from that of *B* shows $(B - A)$.

EXAMPLE Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the universal set, and let $A = \{1, 3, 4, 5\}$ and $B = \{2, 4, 5, 6\}$ be its subsets.

Then, $A \cap B = \{4, 5\}$.

We draw the Venn diagram, as shown in the given figure.

Clearly, $(A - B) = \{1, 3\}$ and $(B - A) = \{2, 6\}$.



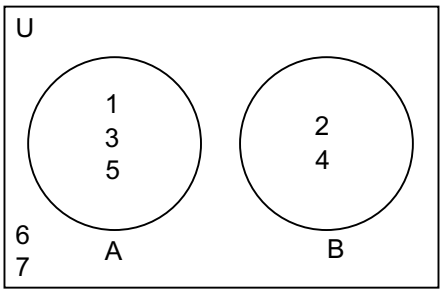
CASE 3 *When two disjoint subsets of a set be given*

In order to represent two disjoint subsets *A* and *B* of the universal set *U*, we draw two disjoint circles within a rectangle.

EXAMPLE Let $U = \{1, 2, 3, 4, 5, 6, 7\}$ be the universal set, and let $A = \{1, 3, 5\}$ and $B = \{2, 4\}$ be two of its disjoint subsets.

Clearly, $A \cap B = \phi$.

So, we may draw the Venn diagram, as shown in the adjoining figure.



Clearly, $A \cap B = \phi$, $(A - B) = \{1, 3, 5\} = A$ and $(B - A) = \{2, 4\} = B$.

$A' = \{2, 4, 6, 7\}$ and $B' = \{1, 3, 5, 6, 7\}$.

CASE 4 *When* $B \subseteq A \subseteq U$

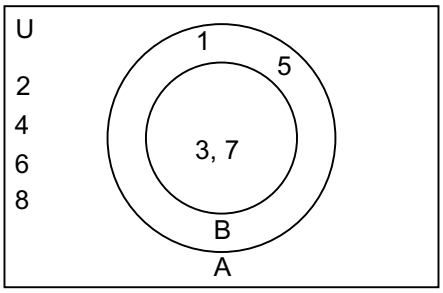
In this case, we draw two concentric circles within a rectangular region.

The inner circle represents *B* and the outer circle represents *A*.

EXAMPLE Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the universal set, and let $A = \{1, 3, 5, 7\}$ and $B = \{3, 7\}$ be its subsets.

Then, clearly $B \subseteq A$.

Now, we may draw the Venn diagram, as shown in the given figure.



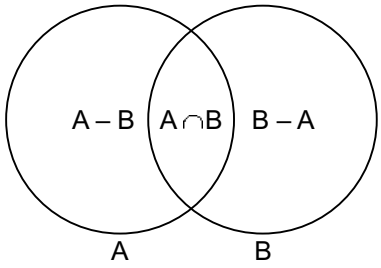
$A \cap B = B = \{3, 7\}$, $A \cup B = A = \{1, 3, 5, 7\}$,

$(A - B) = \{1, 5\}$, $(B - A) = \phi$,

$A' = \{2, 4, 6, 8\}$ and $B' = \{1, 5, 2, 4, 6, 8\}$.

IMPORTANT RESULTS FROM VENN DIAGRAMS

Let *A* and *B* be two intersecting subsets of *U*.



In counting the elements of $(A \cup B)$, the elements of $A \cap B$ are counted twice, once in counting the elements of *A* and second time in counting the elements of *B*.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

If $A \cap B = \phi$, then $n(A \cap B) = 0$ and therefore, in this case, we have

$n(A \cup B) = n(A) + n(B)$.

From the Venn diagram, it is also clear that

(i) $n(A - B) + n(A \cap B) = n(A)$

(ii) $n(B - A) + n(A \cap B) = n(B)$

(iii) $n(A - B) + n(A \cap B) + n(B - A) = n(A \cup B)$.

EXERCISE 1F

1. Let $A = \{a, b, c, e, f\}$, $B = \{c, d, e, g\}$ and $C = \{b, c, f, g\}$ be subsets of the set $U = \{a, b, c, d, e, f, g, h\}$.

Draw Venn diagrams to represent the following sets:

(i) $A \cap B$ (ii) $A \cup (B \cap C)$ (iii) $A - B$

(iv) $B - A$ (v) $A - (B \cap C)$ (vi) $(B - C) \cup (C - B)$

2. Let $A = \{2, 4, 6, 8, 10\}$, $B = \{4, 8, 12, 16\}$ and $C = \{6, 12, 18, 24\}$.

Using Venn diagrams, verify that:

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$

3. Let $A = \{a, e, i, o, u\}$, $B = \{a, d, e, o, v\}$ and $C = \{e, o, t, m\}$.

Using Venn diagrams, verify the following:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. Let $A \subset B \subset U$. Exhibit it in a Venn diagram.

5. Let $A = \{2, 3, 5, 7, 11, 13\}$ and $B = \{5, 7, 9, 11, 15\}$ be subsets of $U = \{2, 3, 5, 7, 9, 11, 13, 15\}$.

Using Venn diagrams, verify that:

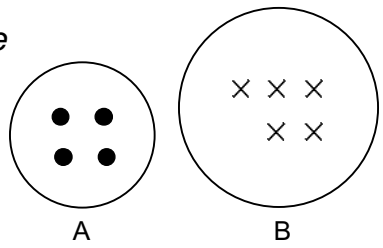
(i) $(A \cup B)' = (A' \cap B')$ (ii) $(A \cap B)' = (A' \cup B')$

6. Using Venn diagrams, show that $(A - B), (A \cap B)$ and $(B - A)$ are disjoint sets, taking $A = \{2, 4, 6, 8, 10, 12\}$ and $B = \{3, 6, 9, 12, 15\}$.

SOME RESULTS DERIVED FROM VENN DIAGRAMS

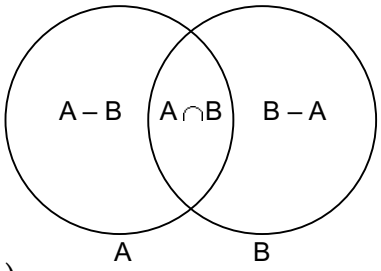
RESULT 1 *When A and B are disjoint sets, then we have*

$$n(A \cup B) = n(A) + n(B).$$



RESULT 2 *For any sets A and B, prove that*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$



PROOF From the given Venn diagram, it is clear that the sets $(A - B), (A \cap B)$ and $(B - A)$ are disjoint and their union is $(A \cup B)$.

$$\therefore \quad n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$$

$$= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B)$$

[adding and subtracting $n(A \cap B)$]

$$= n(A) + n(B) - n(A \cap B)$$

[$\because$ $n(A - B) + n(A \cap B) = n(A)$ and $n(B - A) + n(A \cap B) = n(B)$]

Hence, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Corollary 1 *Prove that* $n(A - B) + n(A \cap B) = n(A)$.

PROOF Clearly, $(A - B)$ and $(A \cap B)$ are disjoint sets and their union is *A*.

$$\therefore \quad n(A - B) + n(A \cap B) = n(A).$$

Corollary 2 *Prove that* $n(B - A) + n(A \cap B) = n(B)$.

PROOF Clearly, $(B - A)$ and $(A \cap B)$ are disjoint sets whose union is *B*.

$$\therefore \quad n(B - A) + n(A \cap B) = n(B).$$

RESULT 3 *For any sets A, B, C prove that*

$$n(A \cup B \cup C) = [n(A) + n(B) + n(C) + n(A \cap B \cap C)]$$

$$- [n(A \cap B) + n(B \cap C) + n(A \cap C)].$$

PROOF We have

$$n(A \cup B \cup C) = n[(A \cup B) \cup C]$$

$$= n(A \cup B) + n(C) - n[(A \cup B) \cap C]$$

$$= \{n(A) + n(B) - n(A \cap B)\} + n(C) - n[(A \cap C) \cup (B \cap C)$$

$$= \{n(A) + n(B) + n(C) - n(A \cap B)\} - \{n(A \cap C) + n(B \cap C)$$

$$- n(A \cap C \cap B \cap C)\}$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= \{n(A) + n(B) + n(C) + n(A \cap B \cap C)\} - \{n(A \cap B) + n(B \cap C) + n(A \cap C)\}.$$

Hence, the result follows.

SUMMARY

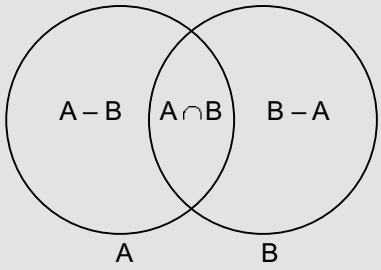
For any sets A, B, C we have:

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

(ii) If $A \cap B = \phi$, then $n(A \cup B) = n(A) + n(B)$.

(iii) $n(A - B) + n(A \cap B) = n(A)$.

(iv) $n(B - A) + n(A \cap B) = n(B)$.



(v) $n(A \cup B \cup C) = \{n(A) + n(B) + n(C) + n(A \cap B \cap C)\} - \{n(A \cap B) + n(B \cap C) + n(A \cap C)\}$.

SOLVED EXAMPLES

EXAMPLE 1 *If A and B are two sets such that $n(A) = 27$, $n(B) = 35$ and $n(A \cup B) = 50$, find $n(A \cap B)$.*

SOLUTION We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad n(A \cap B) = n(A) + n(B) - n(A \cup B) = (27 + 35 - 50) = 12.$$

Hence, $n(A \cap B) = 12$.

EXAMPLE 2 *If A and B are two sets containing 3 and 6 elements respectively, what can be the maximum number of elements in $A \cup B$?*

Find also the minimum number of elements in $A \cup B$.

SOLUTION We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B). \qquad \text{... (i)}$$

CASE 1 From (i), it is clear that $n(A \cup B)$ will be maximum when $n(A \cap B) = 0$.

In that case, $n(A \cup B) = n(A) + n(B) = (3 + 6) = 9$.

$\therefore$ maximum number of elements in $(A \cup B) = 9$.

CASE 2 From (i), it is clear that $n(A \cup B)$ will be minimum when $n(A \cap B)$ is maximum, i.e., when $n(A \cap B) = 3$.

In this case, $n(A \cup B) = n(A) + n(B) - n(A \cap B) = (3 + 6 - 3) = 6$.

$\therefore$ minimum number of elements in $A \cup B = 6$.

EXAMPLE 3 *A survey shows that 73% of the Indians like apples, whereas 65% like oranges. What percentage of Indians like both apples and oranges?*

SOLUTION Let A = set of Indians who like apples

and B = set of Indians who like oranges.

Logarithm

LOGARITHM

If a is a positive real number and $a \neq 1$ such that $a^x = m$, then x is called the logarithm of m to the base a, written as $\log_a m$.

$\therefore \quad a^x = m \iff \log_a m = x.$

LAWS OF LOGARITHM

(i) $\log_a (mn) = (\log_a m) + (\log_a n)$

(ii) $\log_a \left(\frac{m}{n}\right) = (\log_a m) - (\log_a n)$

(iii) $\log_a (m^k) = k \cdot (\log_a m)$

(iv) $\log_a 1 = 0$

(v) $\log_a a = 1$

(vi) $\log_a m = \left(\frac{\log m}{\log a}\right) = \left(\frac{\log_b m}{\log_b a}\right)$

(vii) $\log_a m = \frac{1}{\log_b a}$

COMMON LOGARITHMS

$\log_{10} a$ is called common logarithm, written as $\log a$.

EXAMPLES *(i)* $\log 10 = 1$ *(ii)* $\log 100 = \log (10)^2 = 2$

(iii) $\log \left(\frac{1}{10}\right) = \log (10)^{-1} = -1$, *etc.*

Standard Form of a Decimal

Any number in decimal form expressed as the product of a number between 1 *and* 10, *and an integral power of* 10, *is in standard form.*

Number	Standard form
(i) 27.3	2.73×10^1
(ii) 106.34	1.0634×10^2
(iii) 0.49	4.9×10^{-1}
(iv) 0.08	8×10^{-2}
(v) 0.00037	3.7×10^{-4}

CHARACTERISTIC AND MANTISSA OF A LOGARITHM

log m *consists of two parts:*

(i) the integral part is called *characteristic*,

(ii) the decimal part is called *mantissa.*

Mantissa is always taken as positive, while the characteristic may be positive or negative.

Instead of -2, we write, $\bar{2}$.

REMARK $\bar{2}.7384$ means $(-2 + 0.7384)$.

CHARACTERISTIC OF log *m*

(i) If $m \geq 1$, then its characteristic is one less than the number of digits to the left of the decimal point.

(ii) If $m < 1$, then its characteristic is a negative number whose numerical value is one more than the number of zeros between the decimal point and the first nonzero figure.

ANOTHER METHOD Write the number in standard form: $m \times 10^p$. Then, its characteristic = p.

EXAMPLES

Number	Standard form	Characteristic
5376.4	5.3764×10^3	3
53.764	5.3764×10^1	1
5.3764	5.3764×10^0	0
0.53764	5.3764×10^{-1}	-1
0.00537	5.37×10^{-3}	-3

MANTISSA OF log *m*

We find mantissa from log table.

Position of the decimal point in a number is immaterial for mantissa. Mantissa of log (56), log (.56), log (.0056) is the same.

For finding mantissa of 4385 from log table, we proceed in the row headed by 43 and in this row, we find the number under the column headed by 8. Now, to this number, we add the mean difference headed by 5 and in the same row headed by 43.

Thus, mantissa for 4385 is $(6415 + 5) = .6420$.

REMARK 1 For finding the mantissa of 43 from log table, we find the number in the row headed by 43 under the column 0. It is 0.6335.

REMARK 2 For finding the mantissa of 4 from log table, we find the number in the row headed by 40 and under the column 0. It is 0.6021.

Thus, we have:

(i) $\log 4385 = 3.6420$ (ii) $\log 43.85 = 1.6420$

(iii) $\log 4.385 = 0.6420$ (iv) $\log 0.4385 = \bar{1}.6420$

(v) $\log 0.04385 = \bar{2}.6420$, etc.

ANTILOGARITHM If $\log m = n$, then $m = \text{antilog}\, n$.

EXAMPLE $\log 1000 = 3 \Rightarrow \text{antilog } 3 = 1000$.

Finding Antilogarithm of a Number

For finding the antilog of a number, we use its decimal part and read the antilog table in the same manner as log table.

After finding the number from antilog table, we insert the decimal point as under.

(i) When the characteristic is n, the decimal point is inserted after $(n + 1)$th digit.

(ii) When the characteristic is $\bar{n}$, the decimal point is inserted in such a way that the first significant digit is at the nth place.

EXAMPLES

Number	Antilog
0.2364	1.724
1.4216	26.40
2.6019	399.08
5.2612	182500

Number	Antilog
$\bar{1}.4356$	0.2727
$\bar{2}.0318$	0.01076
$\bar{3}.1459$	0.001399
$\bar{4}.7912$	0.0006183

SOME EXAMPLES

EXAMPLE 1 *Find the value of:*

(i) $\log 3$ *(ii)* $\log 23.45$ *(iii)* $\log 4.17$

(iv) $\log 0.325$ *(v)* $\log 0.0954$ *(vi)* $\log 0.00056$

SOLUTION Using log table, we have

(i) $\log 3 = 0.4771$

(ii) $\log 23.45 = 1.3701$

(iii) $\log 4.17 = 0.6201$

(iv) $\log 0.325 = \bar{1}.5119$

(v) $\log 0.0954 = \bar{2}.9795$

(vi) $\log 0.00056 = \bar{4}.7482$

EXAMPLE 2 *Find the value of:*

(i) antilog 0.654 *(ii) antilog* 1.204

(iii) antilog $\bar{1}.3612$ *(iv) antilog* $\bar{3}.4568$

SOLUTION We have

(i) $\text{antilog}\,(0.654) = 4.508$

(ii) $\text{antilog}\,(1.204) = 16.00$

(iii) $\text{antilog}\,(\bar{1}.3612) = 0.2297$

(iv) $\text{antilog}\,(\bar{3}.4568) = 0.002863$

EXAMPLE 3 *Evaluate* $\dfrac{8492 \times 3.72}{47.8 \times 52.24}$.

SOLUTION Let $x = \dfrac{8492 \times 3.72}{47.8 \times 52.24}$. Then,

$$\log x = \log 8492 + \log 3.72 - \log 47.8 - \log 52.24$$
$$= 3.9290 + 0.5705 - 1.6794 - 1.7180$$
$$= 4.4995 - 3.3974 = 1.1021$$
$$\Rightarrow \quad x = \text{antilog}\,(1.1021) = 12.65.$$

Hence, the value of the given expression is 12.65.

EXAMPLE 4 *Evaluate* $\dfrac{563.4 \times \sqrt[3]{0.4573}}{(6.15)^3}$.

SOLUTION Let $x = \dfrac{563.4 \times \sqrt[3]{0.4573}}{(6.15)^3}$. Then,

$$\log x = \log 563.4 + \frac{1}{3}\log (0.4573) - 3\log (6.15)$$
$$= 2.7508 + \frac{1}{3}(\bar{1}.6602) - 3 \times 0.7889$$
$$= 2.7508 + \frac{1}{3}(-0.3398) - 2.3667$$
$$= 2.7508 - 0.1133 - 2.3667$$
$$= 0.2708$$
$$\Rightarrow \quad x = \text{antilog}\,(0.2708) = 1.865.$$

Hence, the value of the given expression is 1.865.

EXAMPLE 5 *Evaluate* $\sqrt[3]{0.08034}$.

SOLUTION Let $x = \sqrt[3]{0.08034}$. Then,

$$\log x = \frac{1}{3}\log (0.08034).$$
$$= \frac{1}{3}(\bar{2}.9049) = \frac{1}{3}(-2 + 0.9049) = \frac{1}{3}(-1.0951)$$
$$= -0.3650 = -1 + (1 - 0.3650) = \bar{1}.6350$$
$$\Rightarrow \quad x = \text{antilog}\,(\bar{1}.6350) = 0.4315.$$

EXAMPLE 6 *Evaluate* $\sqrt[7]{0.00003587}$.

SOLUTION Let $x = \sqrt[7]{0.00003587}$. Then,

$$\log x = \frac{1}{7}\log (0.00003587)$$
$$= \frac{1}{7}(\bar{5}.5548) = \frac{1}{7}(-5 + 0.5548)$$
$$= \frac{1}{7}(-4.4452) = -0.6350$$
$$= -1 + (1 - 0.6350) = \bar{1}.3650$$

$\Rightarrow \quad x = \text{antilog}\,(\bar{1}.3650) = 0.2317.$

Hence, the required value is 0.2317.

EXAMPLE 7 *Evaluate* $\sqrt{\dfrac{(76.24)^5 \times \sqrt[3]{65}}{(3.2)^7 \times \sqrt{17}}}$.

SOLUTION Let $x = \sqrt{\dfrac{(76.24)^5 \times \sqrt[3]{65}}{(3.2)^7 \times \sqrt{17}}}$. Then,

$$\log x = \frac{1}{2} \cdot \log \left\{ \frac{(76.24)^5 \times (65)^{\frac{1}{3}}}{(3.2)^7 \times (17)^{\frac{1}{2}}} \right\}$$

$$= \frac{1}{2} \cdot \left\{ 5 \log (76.24) + \frac{1}{3} \log 65 - 7 \log (3.2) - \frac{1}{2} \log 17 \right\}$$

$$= \frac{1}{2} \cdot \left\{ 5 \times 1.8822 + \frac{1}{3} \times 1.8129 - 7 \times 0.5051 - \frac{1}{2} \times 1.2304 \right\}$$

$$= \frac{1}{2} \cdot \{9.4110 + 0.6043 - 3.5357 - 0.6152\}$$

$$= \frac{1}{2} \times (10.0153 - 4.1509) = \frac{1}{2} \times 5.8644 = 2.9322$$

$\Rightarrow \quad x = \text{antilog}\,(2.9322) = 855.5.$

Hence, the required value of the given expression is 855.5.

EXERCISE 1

Using log table evaluate the following:

1. $69.13 \times 0.34 \times 0.014$

2. 0.7625×0.000357

3. $358.6 \times 0.078 \times 0.5943$

4. $\dfrac{8.67 \times 99}{1.78}$

5. $(0.09634)^3$

6. $\sqrt[5]{8.0125}$

7. $\sqrt[7]{142.71}$

8. $\dfrac{0.9876 \times (16.42)^2}{(4.567)^{\frac{1}{3}}}$

9. $\dfrac{(6.45)^3 \times (0.00034)^{\frac{1}{3}} \times (981.4)}{(9.37)^2 \times (8.93)^{\frac{1}{4}} \times (0.0617)}$

ANSWERS (EXERCISE 1)

1. 0.3291 **2.** 0.0002723 **3.** 16.63 **4.** 19.19

5. 0.0008941 **6.** 7.517 **7.** 2.032 **8.** 160.4

9. 1963